THE PICARD GROUP OF THE UNIVERSAL MODULI SPACE OF VECTOR BUNDLES ON STABLE CURVES.

ROBERTO FRINGUELLI

Abstract. We construct the moduli stack of properly balanced vector bundles on semistable curves and we determine explicitly its Picard group. As a consequence, we obtain an explicit description of the Picard groups of the universal moduli stack of vector bundles on smooth curves and of the Schmitt's compactification over the stack of stable curves. We prove some results about the gerbe structure of the universal moduli stack over its rigidification by the natural action of the multiplicative group. In particular, we give necessary and sufficient conditions for the existence of Poincaré bundles over the universal curve of an open substack of the rigidification, generalizing a result of Mestrano-Ramanan.

Contents

Introduction.

Let $\mathcal{V}ec^{(s)s}_{r,d,g}$ be the moduli stack of (semi)stable vector bundles of rank r and degree d on smooth curves of genus g. It turns out that the forgetful map $\mathcal{V}ec^{ss}_{r,d,g} \to \mathcal{M}_g$ is universally closed, i.e. it satisfies the existence part of the valuative criterion of properness. Unfortunately, if we enlarge the moduli problem, adding slope-semistable (with respect to the canonical polarization) vector bundles on stable curves, the morphism to the moduli stack $\overline{\mathcal{M}}_g$ of stable curves is not universally closed anymore. There exists two natural ways to make it universally closed. The first one is adding slope-semistable torsion free sheaves and this was done by Pandharipande in [Pan96]. The disadvantage is that such stack, as Faltings has shown

in [Fal96], is not regular if the rank is greater than one. The second approach, which is better for our purposes, is to consider vector bundles on semistable curves: see [Gie84], [Kau05], [NS99] in the case of a fixed irreducible curve with one node, [Cap94], [Mel09] in the rank one case over the entire moduli stack $\overline{\mathcal{M}}_g$ or [Sch04], [TiB98] in the higher rank case over $\overline{\mathcal{M}}_g$. The advantages are that such stacks are regular and the boundary has normal-crossing singularities. Unfortunately, for rank greater than one, we do not have an easy description of the objects at the boundary. We will overcome the problem by constructing a non quasi-compact smooth stack $\overline{\mathcal{V}ec}_{r,d,g}$, parametrizing properly balanced vector bundles on semistable curves (see §1.1 for a precise definition). In some sense, this is the right stacky-generalization in higher rank of the Caporaso's compactification $\overline{J}_{d,g}$ of the universal Jacobian scheme. Moreover it contains some interesting open substacks, like:

- The moduli stack $\mathcal{V}ec_{r,d,g}$ of (not necessarily semistable) vector bundles over smooth curves.
- The moduli stack $\overline{\mathcal{V}ec}^{P(s)s}_{r,d,g}$ of vector bundles such that their push-forwards in the stable model of the curve is a slope-(semi)stable torsion free sheaf.
- The moduli stack $\overline{\mathcal{V}ec}^{H(s)s}_{r,d,g}$ of H-(semi)stable vector bundles constructed by Schmitt in [Sch04].
- The moduli stack of Hilbert-semistable vector bundles (see [TiB98]).

The main result of this paper is computing and giving explicit generators for the Picard groups of the moduli stacks $\mathcal{V}ec_{r,d,g}$ and $\overline{\mathcal{V}ec}_{r,d,g}$ for rank greater than one, generalizing the results in rank one obtained by Melo-Viviani in [MV14], based upon a result of Kouvidakis (see [Kou91]). As a consequence, we will see that there exist natural isomorphisms of Picard groups between $\mathcal{V}ec^{ss}_{r,d,g}$ and $\mathcal{V}ec_{r,d,g}$, among $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$, $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ and $\overline{\mathcal{V}ec}_{r,d,g}$ and between $\overline{\mathcal{V}ec}^{Ps}_{r,d,g}$ and $\overline{\mathcal{V}ec}^{Hs}_{r,d,g}$.

The motivation for this work comes from the study of modular compactifications of the moduli stack $\mathcal{V}ec^{ss}_{r,d,g}$ and the coarse moduli space $U_{r,d,g}$ of semistable vector bundles on smooth curves from the point of view of the log-minimal model program (LMMP). One would like to mimic the so called Hassett-Keel program for the moduli space $\overline{M}_g$ of stable curves, which aims at giving a modular interpretation to the every step of the LMMP fot $\overline{M}_g$. In other words, the goal is to construct compactifications of the universal moduli space of semistable vector bundles over each step of the minimal model program for $\overline{M}_g$. In the rank one case, the conjectural first two steps of the LMMP for the Caporaso's compactification $\overline{J}_{d,g}$ have been described by Bini-Felici-Melo-Viviani in [BFMV14]. From the stacky point of view, the first step (resp. the second step) is constructed as the compactified Jacobian over the Schubert's moduli stack $\overline{\mathcal{M}}^{ps}_g$ of pseudo-stable curves (resp. over the Hyeon-Morrison's moduli stack $\overline{\mathcal{M}}^{wp}_g$ of weakly-pseudo-stable curves). In higher rank, the conjectural first step of the LMMP for the Pandharipande's compactification $\widetilde{U}_{r,d,g}$ has been described by Grimes in [Gri]: using the torsion free approach, he constructs a compactification $\widetilde{U}^{ps}_{r,d,g}$ of the moduli space of slope-semistable vector bundles over $\overline{M}^{ps}_g$. In order to construct birational compact models for the Pandharipande compactification of $U_{r,d,g}$, it is useful to have an explicit description of its rational Picard group which naturally embeds into the rational Picard group of the moduli stack $\mathcal{TF}^{ss}_{r,d,g}$ of slope-semistable torsion free sheaves over stable curves. Indeed our first idea was to study directly the Picard group of $\mathcal{TF}^{ss}_{r,d,g}$. For technical difficulties due to the fact that such stack is not smooth, we have preferred to study first $\overline{\mathcal{V}ec}_{r,d,g}$, whose Picard group contains $\mathrm{Pic}(\mathcal{TF}^{ss}_{r,d,g})$, and we plan to give a description of $\mathcal{TF}^{ss}_{r,d,g}$ in a subsequent paper.

In Section 1, we introduce and study our main object: *the universal moduli stack $\overline{\mathcal{V}ec}_{r,d,g}$ of properly balanced vector bundles of rank r and degree d on semistable curves of arithmetic genus g*. We will show that it is an irreducible smooth Artin stack of dimension $(r^2+3)(g-1)$. The stacks of the above list are contained in $\overline{\mathcal{V}ec}_{r,d,g}$ in the following way

$$\begin{array}{ccccccccc} \overline{\mathcal{V}ec}^{Ps}_{r,d,g} & \subset & \overline{\mathcal{V}ec}^{Hs}_{r,d,g} & \subset & \overline{\mathcal{V}ec}^{Hss}_{r,d,g} & \subset & \overline{\mathcal{V}ec}^{Pss}_{r,d,g} & \subset & \overline{\mathcal{V}ec}_{r,d,g} \\ \cup & & & & \cup & & & & \cup \\ \mathcal{V}ec^{s}_{r,d,g} & & \subset & & \mathcal{V}ec^{ss}_{r,d,g} & & \subset & & \mathcal{V}ec_{r,d,g}. \end{array} \tag{0.0.1}$$

The stack $\overline{\mathcal{V}ec}_{r,d,g}$ is endowed with a morphism $\overline{\phi}_{r,d}$ to the stack $\overline{\mathcal{M}}_g$ which forgets the vector bundle and sends a curve to its stable model. Moreover, it has a structure of $\mathbb{G}_m$-stack, since the group $\mathbb{G}_m$ naturally injects into

the automorphism group of every object as multiplication by scalars on the vector bundle. Therefore, $\overline{\mathcal{V}ec}_{r,d,g}$ becomes a $\mathbb{G}_m$-gerbe over the $\mathbb{G}_m$-rigidification $\overline{\mathcal{V}}_{r,d,g} := \overline{\mathcal{V}ec}_{r,d,g} \,/\!\!/ \mathbb{G}_m$. Let $\nu_{r,d} : \overline{\mathcal{V}ec}_{r,d,g} \to \overline{\mathcal{V}}_{r,d,g}$ be the rigidification morphism. Analogously, the open substacks in (0.0.1) are $\mathbb{G}_m$-gerbes over their rigidifications

$$\begin{array}{ccccccccc} \overline{\mathcal{V}}^{Ps}_{r,d,g} & \subset & \overline{\mathcal{V}}^{Hs}_{r,d,g} & \subset & \overline{\mathcal{V}}^{Hss}_{r,d,g} & \subset & \overline{\mathcal{V}}^{Pss}_{r,d,g} & \subset & \overline{\mathcal{V}}_{r,d,g} \\ \cup & & & & \cup & & & & \cup \\ \mathcal{V}^{s}_{r,d,g} & & \subset & & \mathcal{V}^{ss}_{r,d,g} & & \subset & & \mathcal{V}_{r,d,g}. \end{array} \tag{0.0.2}$$

The inclusions (0.0.1) and (0.0.2 give us the following commutative diagram of Picard groups:

$$\begin{tikzcd}
 & \mathrm{Pic}\left(\overline{\mathcal{V}ec}_{r,d,g}\right) \arrow[rr, two heads] \arrow[dddd, two heads] & & \mathrm{Pic}\left(\overline{\mathcal{V}ec}^{Pss}_{r,d,g}\right) \arrow[rr, two heads] \arrow[dd, two heads] & & \mathrm{Pic}\left(\overline{\mathcal{V}ec}^{Ps}_{r,d,g}\right) \arrow[dd, two heads] \\
\mathrm{Pic}\left(\overline{\mathcal{V}}_{r,d,g}\right) \arrow[ur, hook] \arrow[rr, two heads] \arrow[dddd, two heads] & & \mathrm{Pic}\left(\overline{\mathcal{V}}^{Pss}_{r,d,g}\right) \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, two heads] & & \mathrm{Pic}\left(\overline{\mathcal{V}}^{Ps}_{r,d,g}\right) \arrow[ur, hook] \arrow[dd, two heads] & \\
 & & & \mathrm{Pic}\left(\overline{\mathcal{V}ec}^{Hss}_{r,d,g}\right) \arrow[rr, two heads] \arrow[dd, two heads] & & \mathrm{Pic}\left(\overline{\mathcal{V}ec}^{Hs}_{r,d,g}\right) \arrow[dd, two heads] \\
 & & \mathrm{Pic}\left(\overline{\mathcal{V}}^{Hss}_{r,d,g}\right) \arrow[ur, hook] \arrow[rr, two heads] \arrow[dd, two heads] & & \mathrm{Pic}\left(\overline{\mathcal{V}}^{Hs}_{r,d,g}\right) \arrow[ur, hook] \arrow[dd, two heads] & \\
 & \mathrm{Pic}\left(\mathcal{V}ec_{r,d,g}\right) \arrow[rr, two heads] & & \mathrm{Pic}\left(\mathcal{V}ec^{ss}_{r,d,g}\right) \arrow[rr, two heads] & & \mathrm{Pic}\left(\mathcal{V}ec^{s}_{r,d,g}\right) \\
\mathrm{Pic}\left(\mathcal{V}_{r,d,g}\right) \arrow[ur, hook] \arrow[rr, two heads] & & \mathrm{Pic}\left(\mathcal{V}^{ss}_{r,d,g}\right) \arrow[ur, hook] \arrow[rr, two heads] & & \mathrm{Pic}\left(\mathcal{V}^{s}_{r,d,g}\right) \arrow[ur, hook] &
\end{tikzcd} \tag{0.0.3}$$

where the diagonal maps are the inclusions induced by the rigidification morphisms, while the vertical and horizontal ones are the restriction morphisms, which are surjective because we are working with smooth stacks. We will prove that the Picard groups of diagram (0.0.3) are generated by the boundary line bundles and the tautological line bundles, which are defined in Section 2.

In the same section we also describe the irreducible components of the boundary divisor $\overline{\mathcal{V}ec}_{r,d,g}\backslash\mathcal{V}ec_{r,d,g}$. Obviously the boundary is the pull-back via the morphism $\overline{\phi}_{r,d} : \overline{\mathcal{V}ec}_{r,d,g} \to \overline{\mathcal{M}}_g$ of the boundary of $\overline{\mathcal{M}}_g$. It is known that $\overline{\mathcal{M}}_g\backslash\mathcal{M}_g = \bigcup_{i=0}^{\lfloor g/2\rfloor} \delta_i$, where δ_0 is the irreducible divisor whose generic point is an irreducible curve with just one node and, for $i \neq 0$, δ_i is the irreducible divisor whose generic point is the stable curve with two irreducible smooth components of genus i and $g-i$ meeting in one point. In Proposition 2.6.2, we will prove that $\widetilde{\delta}_i := \overline{\phi}^*_{r,d}(\delta_i)$ is irreducible if $i = 0$ and, otherwise, decomposes as $\bigcup_{j\in J_i} \widetilde{\delta}_i^j$, where J_i is a set of integers depending on i and $\widetilde{\delta}_i^j$ are irreducible divisors. Such $\widetilde{\delta}_i^j$ will be called *boundary divisors.* For special values of i and j, the corresponding boundary divisor will be called *extremal boundary divisor.* The boundary divisors which are not extremal will be called *non-extremal boundary divisors* (for a precise description see §2.6). By smoothness of $\overline{\mathcal{V}ec}_{r,d,g}$, the divisors $\{\widetilde{\delta}_i^j\}$ give us line bundles. We will call them *boundary line bundles* and we will denote them with $\{\mathcal{O}(\widetilde{\delta}_i^j)\}$. We will say that $\mathcal{O}(\widetilde{\delta}_i^j)$ is a *(non)-extremal boundary line bundle* if $\widetilde{\delta}_i^j$ is a (non)-extremal boundary divisor. The irreducible components of the boundary of $\overline{\mathcal{V}}_{r,d,g}$ are the divisors $\nu_{r,d}(\widetilde{\delta}_i^j)$. The associated line bundles are called boundary line bundles of $\overline{\mathcal{V}}_{r,d,g}$. We will denote with the the same symbols used for $\overline{\mathcal{V}ec}_{r,d,g}$ the boundary divisors and the associated boundary line bundles on $\overline{\mathcal{V}}_{r,d,g}$.

In §2.7 we define the *tautological line bundles.* They are defined as determinant of cohomology and as Deligne pairing (see §2.2 for the definition and basic properties) of particular line bundles along the universal curve $\overline{\pi} : \overline{Vec}_{r,d,g,1} \to \overline{\mathcal{V}ec}_{r,d,g}$. More precisely they are

$$\begin{aligned} K_{1,0,0} &:= \langle \omega_{\overline{\pi}}, \omega_{\overline{\pi}} \rangle, \\ K_{0,1,0} &:= \langle \omega_{\overline{\pi}}, \det \mathcal{E} \rangle, \\ K_{-1,2,0} &:= \langle \det \mathcal{E}, \det \mathcal{E} \rangle, \\ \Lambda(m,n,l) &:= d_{\overline{\pi}}(\omega_{\overline{\pi}}^m \otimes (\det \mathcal{E})^n \otimes \mathcal{E}^l). \end{aligned}$$

where $\omega_{\overline{\pi}}$ is the relative dualizing sheaf for π and $\mathcal{E}$ is the universal vector bundle on $\overline{Vec}_{r,d,g,1}$. Following the same strategy of Melo-Viviani in [MV14], based upon the work of Mumford in [Mum83], we apply

Grothendieck-Riemann-Roch theorem to the morphism $\pi : \overline{Vec}_{r,d,g,1} \to \overline{\mathcal{V}ec}_{r,d,g}$ in order to compute the relations among the tautological line bundles in the rational Picard group. In particular, in Theorem 2.7.1 we prove that all tautological line bundles can be expressed in the (rational) Picard group of $\overline{\mathcal{V}ec}_{r,d,g}$ in terms of $\Lambda(1,0,0)$, $\Lambda(0,1,0)$, $\Lambda(1,1,0)$, $\Lambda(0,0,1)$ and the boundary line bundles.

Finally we can now state the main results of this paper. In Section 3, we prove that all Picard groups on diagram (0.0.3) are free and generated by the tautological line bundles and the boundary line bundles. More precisely, we have the following.

Theorem A. *Assume $g \geq 3$ and $r \geq 2$.*

(i) The Picard groups of $\mathcal{V}ec_{r,d,g}$, $\mathcal{V}ec^{ss}_{r,d,g}$, $\mathcal{V}ec^{s}_{r,d,g}$ are freely generated by $\Lambda(1,0,0)$, $\Lambda(1,1,0)$, $\Lambda(0,1,0)$ and $\Lambda(0,0,1)$.

(ii) The Picard groups of $\overline{\mathcal{V}ec}_{r,d,g}$, $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$, $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$ are freely generated by $\Lambda(1,0,0)$, $\Lambda(1,1,0)$, $\Lambda(0,1,0)$, $\Lambda(0,0,1)$ and the boundary line bundles.

(iii) The Picard groups of $\overline{\mathcal{V}ec}^{Ps}_{r,d,g}$, $\overline{\mathcal{V}ec}^{Hs}_{r,d,g}$ are freely generated by $\Lambda(1,0,0)$, $\Lambda(1,1,0)$, $\Lambda(0,1,0)$, $\Lambda(0,0,1)$ and the non-extremal boundary line bundles.

Let $v_{r,d,g}$ and $n_{r,d}$ be the numbers defined in the Notations 0.0.1 below. Let α and β be (not necessarily unique) integers such that $\alpha(d+1-g)+\beta(d+g-1) = -\frac{1}{n_{r,d}} \cdot \frac{v_{1,d,g}}{v_{r,d,g}}(d+r(1-g))$. We set

$$\Xi := \Lambda(0,1,0)^{\frac{d+g-1}{v_{1,d,g}}} \otimes \Lambda(1,1,0)^{-\frac{d-g+1}{v_{1,d,g}}}, \quad \Theta := \Lambda(0,0,1)^{\frac{r}{n_{r,d}} \cdot \frac{v_{1,d,g}}{v_{r,d,g}}} \otimes \Lambda(0,1,0)^{\alpha} \otimes \Lambda(1,1,0)^{\beta}.$$

Theorem B. *Assume $g \geq 3$ and $r \geq 2$.*

(i) The Picard groups of $\mathcal{V}_{r,d,g}$, $\mathcal{V}^{ss}_{r,d,g}$, $\mathcal{V}^{s}_{r,d,g}$ are freely generated by $\Lambda(1,0,0)$, Ξ and Θ.

(ii) The Picard groups of $\overline{\mathcal{V}}_{r,d,g}$, $\overline{\mathcal{V}}^{Pss}_{r,d,g}$, $\overline{\mathcal{V}}^{Hss}_{r,d,g}$ are freely generated by $\Lambda(1,0,0)$, Ξ, Θ and the boundary line bundles.

(iii) The Picard groups of $\overline{\mathcal{V}}^{Ps}_{r,d,g}$ and $\overline{\mathcal{V}}^{Hs}_{r,d,g}$ are freely generated by $\Lambda(1,0,0)$, Ξ, Θ and the non-extremal boundary line bundles.

If we remove the word "freely" from the assertions, the above theorems hold also in the genus two case. This will be shown in appendix A, together with an explicit description of the relations among the generators.

We sketch the strategy of the proofs of the Theorems A and B. First, in §3.1, we will prove that the boundary line bundles are linearly independent. Since the stack $\overline{\mathcal{V}ec}_{r,d,g}$ is smooth and it contains quasi-compact open substacks which are "large enough" and admit a presentation as quotient stacks, we have a natural exact sequence of groups

$$\bigoplus_{i=0,\dots,\lfloor g/2\rfloor} \oplus_{j\in J_i} \langle \mathcal{O}(\widetilde{\delta}^j_i)\rangle \longrightarrow \mathrm{Pic}(\overline{\mathcal{V}ec}_{r,d,g}) \to \mathrm{Pic}(\mathcal{V}ec_{r,d,g}) \longrightarrow 0 \tag{0.0.4}$$

In Theorem 3.1.1, we show that such sequence is also left exact. The strategy that we will use is the same as the one of Arbarello-Cornalba for $\overline{\mathcal{M}}_g$ in [AC87] and the generalization for $\overline{\mathcal{J}ac}_{d,g}$ done by Melo-Viviani in [MV14]. More precisely, we will construct morphisms $B \to \overline{\mathcal{V}ec}_{r,d,g}$ from irreducible smooth projective curves B and we show that the intersection matrix between these test curves and the boundary line bundles on $\overline{\mathcal{V}ec}_{r,d,g}$ is non-degenerate.

Furthermore, since the homomorphism of Picard groups induced by the rigidification morphism $\nu_{r,d}$: $\overline{\mathcal{V}ec}_{r,d,g} \to \overline{\mathcal{V}}_{r,d,g}$ is injective and it sends the boundary line bundles of $\overline{\mathcal{V}}_{r,d,g}$ to the boundary line bundles of $\overline{\mathcal{V}ec}_{r,d,g}$, we see that also the boundary line bundles in the rigidification $\overline{\mathcal{V}}_{r,d,g}$ are linearly independent (see Corollary 3.1.9). In other words we have an exact sequence:

$$0 \longrightarrow \bigoplus_{i=0,\dots,\lfloor g/2\rfloor} \oplus_{j\in J_i} \langle \mathcal{O}(\widetilde{\delta}^j_i)\rangle \longrightarrow \mathrm{Pic}(\overline{\mathcal{V}}_{r,d,g}) \longrightarrow \mathrm{Pic}(\mathcal{V}_{r,d,g}) \longrightarrow 0. \tag{0.0.5}$$

We will show that the sequence (0.0.4, (resp. (0.0.5), remains exact if we replace the middle term with the Picard group of $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ (resp. $\overline{\mathcal{V}}^{Pss}_{r,d,g}$) or $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$ (resp. $\overline{\mathcal{V}}^{Hss}_{r,d,g}$). This reduces the proof of Theorem A(ii) (resp. of Theorem B(ii)) to proving the Theorem A(i) (resp. to Theorem B(i)). While for the stacks $\overline{\mathcal{V}ec}^{Ps}_{r,d,g}$ and $\overline{\mathcal{V}ec}^{Hs}_{r,d,g}$ (resp. $\overline{\mathcal{V}}^{Ps}_{r,d,g}$ and $\overline{\mathcal{V}}^{Hs}_{r,d,g}$) the sequence (0.0.4) (resp. (0.0.5)) is exact if we remove the extremal

boundary line bundles. This reduces the proof of Theorem A(iii) (resp. of Theorem B(iii)) to proving the Theorem A(i) (resp. the Theorem B(i)).

The stack $\mathcal{V}ec_{r,d,g}$ admits a natural map *det* to the universal Jacobian stack $\mathcal{J}ac_{d,g}$, which sends a vector bundle to its determinant line bundle. The morphism is smooth and the fiber over a polarized curve $(C, \mathcal{L})$ is the irreducible moduli stack $\mathcal{V}ec_{=\mathcal{L},C}$ of pairs $(\mathcal{E}, \varphi)$, where $\mathcal{E}$ is a vector bundle on C and φ is an isomorphism between $\det \mathcal{E}$ and $\mathcal{L}$ (for more details see §2.5). Hoffmann in [Hof12] showed that the pull-back to $\mathcal{V}ec_{=\mathcal{L},C}$ of the tautological line bundle $\Lambda(0,0,1)$ on $\overline{\mathcal{V}ec}_{r,d,g}$ freely generates $\mathrm{Pic}\,(\mathcal{V}ec_{=\mathcal{L},C})$ (see Theorem 2.5.1). Moreover, as Melo-Viviani have shown in [MV14], the tautological line bundles $\Lambda(1,0,0)$, $\Lambda(1,1,0)$, $\Lambda(0,1,0)$ freely generate the Picard group of $\mathcal{J}ac_{d,g}$ (see Theorem 2.4.1). Since the the Picard groups of $\mathcal{V}ec_{r,d,g}$, $\mathcal{V}ec^{ss}_{r,d,g}$, $\mathcal{V}ec^{s}_{r,d,g}$ are isomorphic (see Lemma 3.1.5), Theorem A(i) (and so Theorem A) is equivalent to prove that we have an exact sequence of groups

$$0 \longrightarrow \mathrm{Pic}(\mathcal{J}ac_{d,g}) \longrightarrow \mathrm{Pic}(\mathcal{V}ec^{ss}_{r,d,g}) \longrightarrow \mathrm{Pic}(\mathcal{V}ec^{ss}_{=\mathcal{L},C}) \longrightarrow 0 \tag{0.0.6}$$

where the first map is the pull-back via the determinant morphism and the second one is the restriction along a fixed geometric fiber. We will prove this in §3.2. If we were working with schemes, this would follow from the so-called seesaw principle: if we have a proper flat morphism of varieties with integral geometric fibers then a line bundle on the source is the pull-back of a line bundle on the target if and only if it is trivial along any geometric fiber. We generalize this principle to stacks admitting a proper good moduli space (see Appendix B) and we will use this fact to prove the exactness of (0.0.6).
In §3.3, we use the Leray spectral sequence for the lisse-étale sheaf $\mathbb{G}_m$ with respect to the rigidification morphism $\nu_{r,d} : \mathcal{V}ec_{r,d,g} \longrightarrow \mathcal{V}_{r,d,g}$, in order to conclude the proof of Theorem B. Moreover we will obtain, as a consequence, some interesting results about the properties of $\overline{\mathcal{V}}_{r,d,g}$ (see Proposition 3.3.4). In particular we will show that the rigidified universal curve $\mathcal{V}_{r,d,g,1} \to \mathcal{V}_{r,d,g}$ admits a universal vector bundle over an open substack of $\mathcal{V}_{r,d,g}$ if and only if the integers $d + r(1-g)$, $r(d+1-g)$ and $r(2g-2)$ are coprime, generalizing the result of Mestrano-Ramanan ([MR85, Corollary 2.9]) in the rank one case.

The paper is organized in the following way. In Section 1, we define and study the moduli stack $\overline{\mathcal{V}ec}_{r,d,g}$ of properly balanced vector bundles on semistable curves. In §1.1, we give the definition of a properly balanced vector bundle on a semistable curve and we study the properties. In §1.2 we prove that the moduli stack $\overline{\mathcal{V}ec}_{r,d,g}$ is algebraic. In §1.3 we focus on the existence of good moduli spaces for an open substack of $\overline{\mathcal{V}ec}_{r,d,g}$, following the Schmitt's construction. In §1.4 we list some properties of our stacks and we introduce the rigidified moduli stack $\overline{\mathcal{V}}_{r,d,g}$. We will use the deformation theory of vector bundles on nodal curves for study the local structure of $\overline{\mathcal{V}ec}_{r,d,g}$ (see §1.5). In Section 2, we resume some basic facts about the Picard group of a stack. In §2.1 we explain the relations between the Picard group and the Chow group of divisors of stacks. We illustrate how to construct line bundles on moduli stacks using the determinant of cohomology and the Deligne pairing (see §2.2). Then we recall the computation of the Picard group of the stack $\overline{\mathcal{M}}_g$, resp. $\mathcal{J}ac_{d,g}$, resp. $\mathcal{V}ec_{=\mathcal{L},C}$ (see §2.3, resp. §2.4, resp. §2.5). In §2.6 we describe the boundary divisors of $\overline{\mathcal{V}ec}_{r,d,g}$, while in §2.7 we define the tautological line bundles and we study the relations among them. Finally, in Section 3, as explained before, we prove Theorems A and B. The genus two case will be treated separately in the Appendix A. In Appendix B, we recall the definition of a good moduli space for a stack and we develop, following the strategy adopted by Brochard in [Bro12, Appendix], a base change cohomology theory for stacks admitting a proper good moduli space.

Acknowledgements: The author would like to thank his advisor Filippo Viviani, for introducing the author to the problem, for his several suggests and comments without which this work would not have been possible.

Notations.

0.0.1. Let $g \geq 2$, $r \geq 1$, d be integers. We will denote with g the arithmetic genus of the curves, d the degree of the vector bundles and r their rank. Given two integers s, t we will denote with (s,t) the greatest common divisor of s and t. We will set

$$n_{r,d} := (r,d),\ v_{r,d,g} := \left(\frac{d}{n_{r,d}} + \frac{r}{n_{r,d}}(1-g),\, d+1-g,\, 2g-2\right),\ k_{r,d,g} := \frac{2g-2}{(2g-2, d+r(1-g))}.$$

Given a rational number q, we denote with $\lfloor q \rfloor$ the greatest integer such that $\lfloor q \rfloor \leq q$ and with $\lceil q \rceil$ the lowest integer such that $q \leq \lceil q \rceil$.

0.0.2. We will work with the category Sch/k of (not necessarily noetherian) schemes over an algebraically closed field k of characteristic 0. When we say commutative, resp. cartesian, diagram of stacks we will intend in the 2-categorical sense. We will implicitly assume that all the sheaves are sheaves for the site lisse-étale, or equivalently for the site lisse-lisse champêtre (see [Bro, Appendix A.1]).
The choice of characteristic is due to the fact the explicit computation of the Picard group of $\overline{\mathcal{M}}_g$ is known to be true only in characteristic 0 (if $g \geq 3$). Also the computation of $\mathcal{J}ac_{d,g}$ in [MV14] is unknown in positive characteristic, because its computation is based upon a result of Kouvidakis in [Kou91] which is proved over the complex numbers. If these two results could be extended to arbitrary characteristics then also our results would automatically extend.

1. The universal moduli space $\overline{\mathcal{V}ec}_{r,d,g}$.

Here we introduce the moduli stack of properly balanced vector bundles on semistable curves. Before giving the definition, we need to define and study the objects which are going to be parametrized.

Definition 1.0.1. A *stable* (resp. *semistable*) curve C over k is a projective connected nodal curve over k such that any rational smooth component intersects the rest of the curve in at least 3 (resp. 2) points. A *family of (semi)stable curves over a scheme* S is a proper and flat morphism $C \to S$ whose geometric fibers are (semi)stable curves. A *vector bundle on a family of curves* $C \to S$ is a coherent S-flat sheaf on C which is a vector bundle on any geometric fiber.

To any family $C \to S$ of semistable curves, we can associate a new family $C^{st} \to S$ of stable curves and an S-morphism $\pi : C \to C^{st}$, which, for any geometric fiber over S, is the stabilization morphism, i.e. it contracts the rational smooth subcurves intersecting the rest of the curve in exactly 2 points. We can construct this taking the S-morphism $\pi : C \to \mathbb{P}(\omega_{C/S}^{\otimes 3})$ associated to the relative dualizing sheaf of $C \to S$ and calling C^{st} the image of C through π.

Definition 1.0.2. Let C be a semistable curve over k and Z be a non-trivial subcurve. We set $Z^c := \overline{C \backslash Z}$ and $k_Z := |Z \cap Z^c|$. Let $\mathcal{E}$ be a vector bundle over C. If $C_1, \dots C_n$ are the irreducible components of C, we call *multidegree* of $\mathcal{E}$ the n-tuple $(\deg \mathcal{E}_{C_1}, \dots, \deg \mathcal{E}_{C_n})$ and *total degree* of $\mathcal{E}$ the integer $d := \sum \deg \mathcal{E}_{C_i}$.

With abuse of notation we will write $\omega_Z := \deg(\omega_C|_Z) = 2g_Z - 2 + k_Z$, where ω_C is the dualizing sheaf and $g_Z := 1 - \chi(\mathcal{O}_Z)$. If $\mathcal{E}$ is a vector bundle over a family of semistable curves $C \to S$, we will set $\mathcal{E}(n) := \mathcal{E} \otimes \omega_{C/S}^n$. By the projection formula we have

$$R^i\pi_*\mathcal{E}(n) := R^i\pi_*(\mathcal{E} \otimes \omega_{C/S}^n) \cong R^i\pi_*(\mathcal{E}) \otimes \omega_{C^{st}/S}^n$$

where π is the stabilization morphism.

1.1. **Properly balanced vector bundles.**

We recall some definitions and results from [Kau05], [Sch04] and [NS99].

Definition 1.1.1. A *chain of rational curves* (or *rational chain*) R is a connected projective nodal curve over k whose associated graph is a path and whose irreducible components are rational. The *lenght* of R is the number of irreducible components.

Let $R_1, \dots, R_k$ be the irreducible components of a chain of rational curves R, labeled in the following way: $R_i \cap R_j \neq \emptyset$ if and only if $|i-j| \leq 1$. For $1 \leq i \leq k-1$ let $x_i := R_i \cap R_{i+1}$ be the nodal points and $x_0 \in R_1$, $x_k \in R_k$ closed points different from x_1 and x_{k-1}. Let $\mathcal{E}$ be a vector bundle on R of rank r. By [TiB91, Proposition 3.1], any vector bundle $\mathcal{E}$ over a chain of rational curves R decomposes in the following way

$$\mathcal{E} \cong \bigoplus_{j=1}^{r} \mathcal{L}_j, \text{ where } \mathcal{L}_j \text{ is a line bundle for any } j = 1, \dots, r.$$

Using these notations we can give the following definitions.

Definition 1.1.2. Let $\mathcal{E}$ be a vector bundle of rank r on a rational chain R of lenght k.

- $\mathcal{E}$ is *positive* if $\deg \mathcal{L}_{j|R_i} \geq 0$ for any $j \in \{1, \dots, r\}$ and $i \in \{1, \dots, k\}$,

- $\mathcal{E}$ is *strictly positive* if $\mathcal{E}$ is positive and for any $i \in \{1, \dots, k\}$ there exists $j \in \{1, \dots, r\}$ such that $\deg \mathcal{L}_{j|R_i} > 0$,
- $\mathcal{E}$ is *stricly standard* if $\mathcal{E}$ is strictly positive and $\deg \mathcal{L}_{j|R_i} \leq 1$ for any $j \in \{1, \dots, r\}$ and $i \in \{1, \dots, k\}$.

Definition 1.1.3. Let R be a chain of rational curves over k and $R_1, \dots, R_k$ its irreducible components. A strictly standard vector bundle $\mathcal{E}$ of rank r over R is called *admissible*, if one of the following equivalent conditions (see [NS99, Lemma 2] or [Kau05, Lemma 3.3]) holds:

- $h^0(R, \mathcal{E}(-x_0)) = \sum \deg \mathcal{E}_{R_i} = \deg \mathcal{E}$,
- $H^0(R, \mathcal{E}(-x_0 - x_k)) = 0$,
- $\mathcal{E} = \bigoplus_{i=1}^r \mathcal{L}_i$, where $\mathcal{L}_i$ is a line bundle of total degree 0 or 1 for $i = 1, \dots, r$.

Definition 1.1.4. Let C be a semistable curve C over k. The subcurve of all the chains of rational curves will be called *exceptional curve* and will be denoted with C_{exc} and we set $\widetilde{C} := C_{exc}^c$. A connected subcurve R of C_{exc} will be called *maximal rational chain* if there is no rational chain $R' \subset C$ such that $R \subsetneq R'$.

Definition 1.1.5. Let C be a semistable curve and $\mathcal{E}$ be a vector bundle of rank r over C. $\mathcal{E}$ is *(strictly) positive*, resp. *strictly standard*, resp. *admissible vector bundle* if the restriction to any rational chain is (strictly) positive, resp. strictly standard, resp. admissible. Let $C \to S$ be a family of semistable curves with a vector bundle $\mathcal{E}$ of relative rank r. $\mathcal{E}$ is called *(strictly) positive*, resp. *strictly standard*, resp. *admissible vector bundle* if it is (strictly) positive, resp. strictly standard, resp. admissible for any geometric fiber.

Remark 1.1.6. Let $(C, \mathcal{E})$ be a semistable curve with a vector bundle. We have the following sequence of implications: $\mathcal{E}$ is admissible $\Rightarrow$ $\mathcal{E}$ is strictly standard $\Rightarrow$ $\mathcal{E}$ is strictly positive $\Rightarrow$ $\mathcal{E}$ is positive. Moreover if $\mathcal{E}$ is admissible of rank r then any rational chain must be of lenght $\leq r$.

The role of positivity is summarized in the next two propositions.

Proposition 1.1.7. [Sch04, Prop 1.3.1(ii)] *Let $\pi : C' \to C$ be a morphism between semistable curves which contracts only some chains of rational curves. Let $\mathcal{E}$ be a vector bundle on C' positive on the contracted chains. Then $R^i\pi_*(\mathcal{E}) = 0$ for $i > 0$. In particular $H^j(C', \mathcal{E}) = H^j(C, \pi_*\mathcal{E})$ for all j.*

Proposition 1.1.8. *Let $C \to S$ be a family of semistable curves, S locally noetherian scheme and consider the stabilization morphism*

$$\begin{tikzcd} C \arrow[rr, "\pi"] \arrow[dr] & & C^{st} \arrow[dl] \\ & S & \end{tikzcd}$$

Suppose that $\mathcal{E}$ is a positive vector bundle on $C \to S$ and for any point $s \in S$ consider the induced morphism $\pi_{s*} : C_s \to C_s^{st}$. Then

$$\pi_*(\mathcal{E})_{C_s^{st}} = \pi_{s*}(\mathcal{E}_{C_s}).$$

Moreover $\pi_*\mathcal{E}$ is S-flat.

Proof. It follows from [NS99, Lemma 4] and [Sch04, Remark 1.3.6]. $\square$

The next results gives us a useful criterion to check if a vector bundle is strictly positive or not.

Proposition 1.1.9. [Sch04, Proposition 1.3.3]. *Let C be a semistable curve containing the maximal chains $R_1, \dots, R_k$. We set $\widetilde{C}_j := R_j^c$, and let p_1^j, p_2^j be the points where R_j is attached to $\widetilde{C}_j$, for $j = 1, \dots, k$. Suppose that $\mathcal{E}$ is a strictly positive vector bundle on C which satisfies the following conditions:*

(i) $H^1\left(\widetilde{C}_j, \mathcal{I}_{p_1^j, p_2^j}\mathcal{E}_{\widetilde{C}_j}\right) = 0$ *for* $j = 1, \dots, k$.

(ii) The homomorphism

$$H^0\left(\widetilde{C}_j, \mathcal{I}_{p_1^j, p_2^j}\mathcal{E}_{\widetilde{C}_j}\right) \longrightarrow \left(\mathcal{I}_{p_1^j, p_2^j}\mathcal{E}_{\widetilde{C}_j}\right) \Big/ \left(\mathcal{I}^2_{p_1^j, p_2^j}\mathcal{E}_{\widetilde{C}_j}\right)$$

is surjective for $j = 1, \dots, k$.

(iii) For any $x \in \widetilde{C} \backslash \{p_1^j, p_2^j, j = 1, \dots, k\}$, the homomorphism

$$H^0\left(C, \mathcal{I}_{C_{exc}}\mathcal{E}\right) \longrightarrow \mathcal{E}_{\widetilde{C}} \Big/ \left(\mathcal{I}_x^2\mathcal{E}_{\widetilde{C}}\right)$$

is surjective.

(iv) For any $x_1 \neq x_2 \in \widetilde{C} \setminus \{p_1^j, p_2^j, j = 1, \ldots, k\}$*, the evaluation homomorphism*

$$H^0\left(C, \mathcal{I}_{C_{exc}}\mathcal{E}\right) \longrightarrow \mathcal{E}_{\{x_1\}} \oplus \mathcal{E}_{\{x_2\}}$$

is surjective.

Then $\mathcal{E}$ *is generated by global sections and the induced morphism in the Grassmannian*

$$C \hookrightarrow Gr(H^0(C, \mathcal{E}), r)$$

is a closed embedding.

Using [Sch04, Remark 1.3.4], we deduce the following useful criterion

Corollary 1.1.10. *Let* $\mathcal{E}$ *be a vector bundle over a semistable curve* C*.* $\mathcal{E}$ *is strictly positive if and only if there exists* n *big enough such that the vector bundle* $\mathcal{E}(n)$ *is generated by global sections and the induced morphism in the Grassmannian* $C \to Gr(H^0(C, \mathcal{E}(n)), r)$ *is a closed embedding.*

Remark 1.1.11. Let $\mathcal{F}$ be a torsion free sheaf over a nodal curve C. By [Ses82, Huitieme Partie, Proposition 3], the stalk of $\mathcal{F}$ over a nodal point x is of the form

- $\mathcal{O}_{C,x}^{r_0} \oplus \mathcal{O}_{C_1,x}^{r_1} \oplus \mathcal{O}_{C_2,x}^{r_2}$, if x is a meeting point of two irreducible curves C_1 and C_2.
- $\mathcal{O}_{C,x}^{r-a} \oplus m_{C,x}^{a}$, if x is a nodal point belonging to a unique irreducible component.

If $\mathcal{F}$ has uniform rank r (i.e. it has rank r on any irreducible component of C), we can always write the stalk at x in the form $\mathcal{O}_{C,x}^{r-a} \oplus m_{C,x}^{a}$ for some α. In this case we will say that $\mathcal{F}$ is *of type* a *at* x.

Now we are going to describe the properties of an admissible vector bundle. The following proposition (and its proof) is a generalization of [NS99, Proposition 5].

Proposition 1.1.12. *Let* $\mathcal{E}$ *be a vector bundle of rank* r *over a semistable curve* C*, and* $\pi : C \to C^{st}$ *the stabilization morphism, then:*

(i) $\mathcal{E}$ *is admissible if and only if* $\mathcal{E}$ *is strictly positive and* $\pi_*\mathcal{E}$ *is torsion free.*
(ii) Let R *be a maximal chain of rational curves and* $x := \pi(R)$*. If* $\mathcal{E}$ *is admissible then* $\pi_*\mathcal{E}$ *is of type* $\deg \mathcal{E}_R$ *at* x*.*

Proof. Part (i). By hypothesis $\mathcal{E}$ is strictly positive. Let $\widetilde{C}$ be the subcurve of C complementary to the exceptional one. Consider the exact sequence:

$$0 \longrightarrow \mathcal{I}_{\widetilde{C}}\mathcal{E} \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}_{\widetilde{C}} \longrightarrow 0.$$

We can identify $\mathcal{I}_{\widetilde{C}}\mathcal{E}$ with $\mathcal{I}_D\mathcal{E}_{C_{exc}}$, where $D := C_{exc} \cap \widetilde{C}$ with its reduced scheme structure. Then we have:

$$0 \longrightarrow \pi_*(\mathcal{I}_D\mathcal{E}_{C_{exc}}) \longrightarrow \pi_*\mathcal{E} \longrightarrow \pi_*(\mathcal{E}_{\widetilde{C}}).$$

Now $\pi_*(\mathcal{E}_{\widetilde{C}})$ is a torsion-free sheaf and $\pi_*(\mathcal{I}_D\mathcal{E}_{C_{exc}})$ is a torsion sheaf, because its support is D. So $\pi_*\mathcal{E}$ is torsion free if and only if $\pi_*(\mathcal{I}_D\mathcal{E}_{C_{exc}}) = 0$. Let R be a maximal rational chain which intersects the rest of the curve in p and q and $x := \pi(R)$. By definition the stalk of the sheaf $\pi_*(\mathcal{I}_D\mathcal{E}_{C_{exc}})$ at x is the k-vector space $H^0(R, \mathcal{I}_{p,q}\mathcal{E}_R) = H^0(R, \mathcal{E}_R(-p-q))$. Applying this method for any rational chain we have that $\pi_*\mathcal{E}$ is torsion free if and only if for any chain R if a global section s of $\mathcal{E}_R$ vanishes on $R \cap R^c$ then $s \equiv 0$. In particular, if $\mathcal{E}$ is admissible then $\pi_*\mathcal{E}$ is torsion free and $\mathcal{E}$ is strictly positive.
Conversely, suppose that $\pi_*\mathcal{E}$ is torsion free and $\mathcal{E}$ is strictly positive. The definition of admissibility requires that the vector bundle must be strictly standard, so a priori it seems that the viceversa should not be true. However we can easily see that if $\mathcal{E}$ is strictly positive but not strictly standard then there exists a chain R such that $H^0(R, \mathcal{I}_{p,q}\mathcal{E}_R) \neq 0$. So $\pi_*\mathcal{E}$ cannot be torsion free, giving a contradiction. In other words, if $\pi_*\mathcal{E}$ is torsion free and $\mathcal{E}$ is strictly positive then $\mathcal{E}$ is strictly standard. By the above considerations the assertion follows.
Part (ii). Let R be a maximal chain of rational curves. By hypothesis and part (i), $\pi_*\mathcal{E}$ is torsion free and we have an exact sequence:

$$0 \longrightarrow \pi_*\mathcal{E} \longrightarrow \pi_*(\mathcal{E}_{\widetilde{C}}) \longrightarrow R^1\pi_*(\mathcal{I}_D\mathcal{E}_{C_{exc}}) \longrightarrow 0.$$

The sequence is right exact by Proposition 1.1.7. Using the notation of part (i), we have that the stalk of the sheaf $R^1\pi_*(\mathcal{I}_D\mathcal{E}_{C_{exc}})$ at x is the k-vector space $H^1(R, \mathcal{E}_R(-p-q))$. If $\deg(\mathcal{E}_R) = r$ is easy to see that $H^1(R, \mathcal{E}_R(-p-q)) = 0$, thus $\pi_*\mathcal{E}$ is isomorphic to $\pi_*(\mathcal{E}_{\widetilde{C}})$ locally at x. The assertion follows by the fact that

$\pi_*(\mathcal{E}_{\widetilde{C}})$ is a torsion free sheaf of type $\deg(\mathcal{E}_R)=r$ at x. Suppose that $\deg(\mathcal{E}_R)=r-s<r$. Then we must have that $\mathcal{E}_R=\mathcal{O}_R^s\oplus\mathcal{F}$. Using the sequence

$$0\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}_{R^c}\oplus\mathcal{E}_R\longrightarrow\mathcal{E}_{\{p\}}\oplus\mathcal{E}_{\{q\}}\longrightarrow 0$$

we can found a neighbourhood U of x in C^{st} such that $\mathcal{E}_{\pi^{-1}(U)}=\mathcal{O}^s_{\pi^{-1}(U)}\oplus\mathcal{E}'$ reducing to the case $\deg(\mathcal{E}_R)=r$. □

The proposition above has some consequence, which will be useful later. For example in §1.3, where we will prove that a particular subset of the set of admissible vector bundles over a semistable curve is bounded. The following results are generalizations of [NS99, Remark 4].

Corollary 1.1.13.

(i) Let C be a stable curve, $\pi:N\to C$ a partial normalization and $\mathcal{F}_1$, $\mathcal{F}_2$ two vector bundles on N. Then

$$Hom_{\mathcal{O}_N}(\mathcal{F}_1,\mathcal{F}_2)\cong Hom_{\mathcal{O}_C}(\pi_*(\mathcal{F}_1),\pi_*(\mathcal{F}_2)).$$

In particular $\mathcal{F}_1\cong\mathcal{F}_2\iff\pi_*(\mathcal{F}_1)\cong\pi_*(\mathcal{F}_2)$.

(ii) Let C be a semistable curve with an admissible vector bundle $\mathcal{E}$, let R be a subcurve composed only by maximal chains. We set $\widetilde{C}:=R^c$ and D the reduced subscheme $R\cap\widetilde{C}$. Let $\pi:C\to C^{st}$ be the stabilization morphsim and D^{st} be the reduced scheme $\pi(D)$. Then

$$\pi_*\left(\mathcal{I}_R\mathcal{E}_R\right)=\pi_*\left(\mathcal{I}_D\mathcal{E}_{\widetilde{C}}\right)=\mathcal{I}_{D^{st}}\left(\pi_*\mathcal{E}\right).$$

(iii) We set $\widetilde{C}:=C^c_{exc}$. We have that $\pi_*\mathcal{E}$ determines $\mathcal{E}_{\widetilde{C}}$, i.e. consider two pairs $(C,\mathcal{E})$, $(C',\mathcal{E}')$ of semistable curves with admissible vector bundles such that $(C^{st},\pi_*\mathcal{E})\cong(C'^{st},\pi'_*\mathcal{E}')$, then $(\widetilde{C},\mathcal{E}_{\widetilde{C}})\cong(\widetilde{C}',\mathcal{E}'_{\widetilde{C}'})$. Observe that C_{exc} and C'_{exc} can be different.

Proof. Part (i). Adapting the proof of [NS99, Remark 4(ii)] to our more general case, we obtain the assertion. Part (ii). Consider the following exact sequence

$$0\longrightarrow\mathcal{I}_R\mathcal{E}\longrightarrow\mathcal{E}\longrightarrow\mathcal{E}_R\longrightarrow 0.$$

We can identify $\mathcal{I}_R\mathcal{E}$ with $\mathcal{I}_D\mathcal{E}_{\widetilde{C}}$. Applying the left exact functor π_*, we have

$$0\longrightarrow\pi_*\left(\mathcal{I}_D\mathcal{E}_{\widetilde{C}}\right)\longrightarrow\pi_*(\mathcal{E})\longrightarrow\pi_*(\mathcal{E}_R)\longrightarrow 0.$$

The sequence is right exact because $\mathcal{E}$ is positive. Moreover $\pi_*(\mathcal{E}_R)$ is supported at D^{st} and annihilated by $\mathcal{I}_{D^{st}}$. By Proposition 1.1.12(ii), the morphism $\pi_*(\mathcal{E})\longrightarrow\pi_*(\mathcal{E}_R)$ induces an isomorphism of vector spaces at the restriction to D^{st}. This means that $\pi_*\left(\mathcal{I}_D\mathcal{E}_{\widetilde{C}}\right)=\mathcal{I}_{D^{st}}\left(\pi_*\mathcal{E}\right)$.
Part (iii). Suppose that $(C^{st},\pi_*\mathcal{E})\cong(C'^{st},\pi'_*\mathcal{E}')$, i.e. there exist an isomorphism of curves $\psi:C^{st}\to C'^{st}$ and an isomorphism of sheaves $\phi:\pi_*\mathcal{E}\cong\psi^*\pi_*\mathcal{E}'$. By (ii), we have

$$\pi_*(\mathcal{I}_D\mathcal{E}_{\widetilde{C}})\cong\psi^*\pi_*(\mathcal{I}_{D'}\mathcal{E}'_{\widetilde{C}'}).$$

First we observe that $\widetilde{C}$ and $\widetilde{C}'$ are isomorphic and ψ induces an isomorphism $\widetilde{\psi}$ between them, such that

$$\psi^*\pi_*(\mathcal{I}_{D'}\mathcal{E}'_{\widetilde{C}'}).\cong\pi_*(\mathcal{I}_D\widetilde{\psi}^*\mathcal{E}'_{\widetilde{C}'}).$$

Now by (i), we obtain an isomorphism of vector bundles $\mathcal{I}_D\mathcal{E}_{\widetilde{C}}\cong\mathcal{I}_D\widetilde{\psi}^*\mathcal{E}'_{\widetilde{C}'}$. Twisting by $\mathcal{I}_D^{-1}$, we have the assertion. □

Definition 1.1.14. Let $\mathcal{E}$ be a vector bundle of rank r and degree d on a semistable curve C. $\mathcal{E}$ is *balanced* if for any subcurve $Z\subset C$ it satisfies the *basic inequality*:

$$\left|\deg\mathcal{E}_Z-d\frac{\omega_Z}{\omega_C}\right|\leq r\frac{k_Z}{2}.$$

$\mathcal{E}$ is *properly balanced* if is balanced and admissible. If $C\to S$ is a family of semistable curves and $\mathcal{E}$ is a vector bundle of relative rank r for this family, we will call it *(properly) balanced* if is (properly) balanced for any geometric fiber.

Remark 1.1.15. We have several equivalent definitions of balanced vector bundle. We list some which will be useful later:

(i) $\mathcal{E}$ is balanced;

(ii) the basic inequality is satisfied for any subcurve $Z \subset C$ such that Z and Z^c are connected;

(iii) for any subcurve $Z \subset C$ such that Z and Z^c are connected, we have the following inequality
$$\deg \mathcal{E}_Z - d\frac{\omega_Z}{\omega_C} \leq r\frac{k_Z}{2};$$

(iv) for any subcurve $Z \subset C$ such that Z and Z^c are connected, we have the following inequality
$$\frac{\chi(\mathcal{F})}{\omega_Z} \leq \frac{\chi(\mathcal{E})}{\omega_C},$$
where $\mathcal{F}$ is the subsheaf of $\mathcal{E}_Z$ of sections vanishing on $Z \cap Z^c$;

(v) for any subcurve $Z \subset C$ such that Z and Z^c are connected, we have $\chi(\mathcal{G}_Z) \geq 0$, where $\mathcal{G}$ is the vector bundle
$$(\det \mathcal{E})^{\otimes 2g-2} \otimes \omega_{C/S}^{\otimes -d+r(g-1)} \oplus \mathcal{O}_C^{\oplus r(2g-2)-1}.$$

Lemma 1.1.16. *Let $(p : C \to S, \mathcal{E})$ be a vector bundle of rank r and degree d over a family of reduced and connected curves. Suppose that S is locally noetherian. The locus where C is a semistable curve and $\mathcal{E}$ strictly positive, resp. admissible, resp. properly balanced, is open in S.*

Proof. We can suppose that S is noetherian and connected. Suppose that there exists a point $s \in S$ such that the geometric fiber is a properly balanced vector bundle over a semistable curve. It is known that the locus of semistable curves is open on S (see [ACG11, Chap. X, Corollary 6.6]). So we can suppose that $C \to S$ is a family of semistables curves of genus g. Up to twisting by a suitable power of $\omega_{C/S}$ we can assume, by Corollary 1.1.10, that the rational S-morphism
$$i : C \dashrightarrow Gr(p_*\mathcal{E}, r)$$
is a closed embedding over s. By [Kau05, Lemma 3.13], there exists an open neighborhood S' of s such that i is a closed embedding. Equivalently $\mathcal{E}_{S'}$ is strictly positive by Corollary 1.1.10. We denote as usual with $\pi : C \to C^{st}$ the stabilization morphism. By Proposition 1.1.8, the sheaf $\pi_*(\mathcal{E}_{S'})$ is flat over S' and the push-forward commutes with the restriction on the fibers. In particular, it is torsion free at the fiber s, and so there exists an open subset S'' of S' where $\pi_*(\mathcal{E}_{S''})$ is torsion-free over any fiber (see [HL10, Proposition 2.3.1]). By Proposition 1.1.12, $\mathcal{E}_{S''}$ is admissible. Putting everything together, we obtain an open neighbourhood S'' of s such that over any fiber we have an admissible vector bundle over a semistable curve. Let $0 \leq k \leq d$, $0 \leq i \leq g$ be integers. Consider the relative Hilbert scheme
$$Hilb_{C/S''}^{\mathcal{O}_C(1), P(m)=km+1-i}$$
where $\mathcal{O}_C(1)$ is the line bundle induced by the embedding i. We call $H_{k,i}$ the closure of the locus of semistable curves in $Hilb_{C/S''}^{\mathcal{O}_C(1), P(m)=km+1-i}$ and we let $Z_{k,i} \hookrightarrow C \times_{S''} H_{k,i}$ be the universal curve. Consider the vector bundle $\mathcal{G}$ over $C \to S''$ as in Remark 1.1.15(v). Let $\mathcal{G}^{k,i}$ its pull-back on $Z_{k,i}$. The function
$$\chi : h \mapsto \chi(\mathcal{G}_h^{k,i})$$
is locally constant on $H_{k,i}$. Now $\pi : H_{k,i} \to S''$ is projective. So the projection on S'' of the connected components of
$$\coprod_{\substack{0 \leq k \leq d \\ 0 \leq i \leq g}} H_{k,i}$$
such that χ is negative is a closed subscheme. Its complement in S is open and, by Remark 1.1.15(v), it contains s and defines a family of properly balanced vector bundles over semistable curves. $\square$

1.2. The moduli stack of properly balanced vector bundles $\overline{\mathcal{V}ec}_{r,d,g}$.

Now we will introduce our main object of study: *the universal moduli stack $\overline{\mathcal{V}ec}_{r,d,g}$ of properly balanced vector bundles of rank r and degree d on semistable curves of arithmetic genus g*. Roughly speaking, we want a space such that its points are in bijection with the pairs $(C, \mathcal{E})$ where C is a semistable curve on k and $\mathcal{E}$ is a properly balanced vector bundle on C. This subsection is devoted to the construction of such space as Artin stack.

Definition 1.2.1. Let $r \geq 1$, d and $g \geq 2$ be integers. Let $\overline{\mathcal{V}ec}_{r,d,g}$ be the category fibered in groupoids over Sch/k whose objects over a scheme S are the families of semistable curves of genus g with a properly balanced vector bundle of relative total degree d and relative rank r. The arrows between the objects are the obvious cartesian diagrams.

The aim of this subsection is proving the following

Theorem 1.2.2. *$\overline{\mathcal{V}ec}_{r,d,g}$ is an irreducible smooth Artin stack of dimension $(r^2+3)(g-1)$. Furthermore, it admits an open cover $\{\overline{\mathcal{U}}_n\}_{n\in\mathbb{Z}}$ such that $\overline{\mathcal{U}}_n$ is a quotient stack of a smooth noetherian scheme by a suitable general linear group.*

Remark 1.2.3. In the case $r=1$, $\overline{\mathcal{V}ec}_{1,d,g}$ is quasi compact and it corresponds to the compactification of the universal Jacobian over $\overline{\mathcal{M}}_g$ constructed by Caporaso [Cap94] and later generalized by Melo [Mel09]. Following the notation of [MV14], we will set $\overline{\mathcal{J}ac}_{d,g} := \overline{\mathcal{V}ec}_{1,d,g}$.

The proof consists in several steps, following the strategies adopted by Kausz [Kau05] and Wang [Wan]. First, we observe that $\overline{\mathcal{V}ec}_{r,d,g}$ is clearly a stack for the Zariski topology. We now prove that it is a stack also for the fpqc topology (defined in [FGI$^+$05, Section 2.3.2]). With that in mind, we will first prove the following lemma which allows us to restrict to families of semistable curves with properly balanced vector bundles over locally noetherian schemes.

Lemma 1.2.4. *Let $\mathcal{E}$ be a properly balanced vector bundle over a family of semistable curves $p: C\to S$. Suppose that S is affine. Then there exists*

- *a surjective morphism $\phi: S\to T$ where T is a noetherian affine scheme,*
- *a family of semistable curves $C_T\to T$,*
- *a properly balanced vector bundle $\mathcal{E}_T$ over $C_T\to T$,*

such that the pair $(C\to S,\mathcal{E})$ is the pull-back by ϕ of the pair $(C_T\to T,\mathcal{E}_T)$.

Proof. We can write S as a projective limit of affine noetherian k-schemes (S_α). By [**?**, 8.8.2 (ii)] there exists an α, a scheme C_α and a morphism $C_\alpha\to S_\alpha$ such that C is the pull-back of this scheme by $S\to S_\alpha$. By [**?**, 8.10.5 (xii)] and [**?**, 11.2.6 (ii)] we can assume that $C_\alpha\to S_\alpha$ is flat and proper. By [**?**, 8.5.2 (ii)] there exists a coherent sheaf $\mathcal{E}_\alpha$ on C_α such that its pull-back on S is $\mathcal{E}$. Moreover, by [**?**, 11.2.6 (ii)] we may assume that $\mathcal{E}_\alpha$ is S_α-flat. Set $S_\alpha =: T$, $C_\alpha =: C_T$ and $\mathcal{E}_\alpha =: \mathcal{E}_T$. Now the family $C_T\to T$ will be a family of semistable curves. The vector bundle $\mathcal{E}$ is properly balanced because this condition can be checked on the geometric fibers. □

Proposition 1.2.5. *Let $S'\to S$ be an fpqc morphism of schemes, set $S'':=S'\times_S S'$ and π_i the natural projections. Let $(C\to S',\mathcal{E}')\in\overline{\mathcal{V}ec}_{r,d,g}(S')$. Then every descent data*

$$\varphi:\pi_1^*(C\to S',\mathcal{E})\cong\pi_2^*(C\to S',\mathcal{E})$$

is effective.

Proof. First we reduce to the case where S' and S are noetherian schemes. By [**?**, (8.8.2)(ii), (8.10.5)(vi), (8.10.5)(viii) (11.2.6)(ii)] there exists an fpqc morphism of noetherian affine schemes $S_0'\to S_0$ and a morphism $S\to S_0$, such that the diagram

$$\begin{tikzcd} S' \arrow[r] \arrow[d] & S \arrow[d] \\ S_0' \arrow[r] & S_0 \end{tikzcd}$$

is cartesian. By Lemma 1.2.4, there exists a pair $(C_0\to S_0',\mathcal{E}_0')\in\overline{\mathcal{V}ec}_{r,d,g}(S_0')$ such that its pull-back via $S'\to S_0'$ is isomorphic to $(C\to S',\mathcal{E}')\in\overline{\mathcal{V}ec}_{r,d,g}(S')$. By [**?**, (8.8.2)(i), (8.5.2)(i), (8.8.2.4), (8.5.2.4)] we can assume that φ comes from a descent data

$$\varphi_0:\pi_1^*(C\to S_0',\mathcal{E}_0')\cong\pi_2^*(C\to S_0',\mathcal{E}_0').$$

So we can assume that S and S' are noetherian. By the properly balanced condition, up to twisting by some power of the dualizing sheaf, we can suppose that $\det\mathcal{E}'$ is relatively ample on S', in particular φ induces a descent data for $(C\to S',\det\mathcal{E}')$ and this is effective by [FGI$^+$05, Theorem. 4.38]. So there exists a family of curves $C\to S$ such that its pull-back via $S'\to S$ is $C'\to S'$. In particular, $C'\to C$ is an fpqc cover and φ induces a descent data for $\mathcal{E}'$ on $C'\to C$, which is effective by [FGI$^+$05, Theorem. 4.23]. □

Proposition 1.2.6. *Let S be an affine scheme. Let $(C \to S, \mathcal{E})$, $(C' \to S, \mathcal{E}') \in \overline{\mathcal{V}ec}_{r,d,g}(S)$. The contravariant functor*

$$(T \to S) \mapsto Isom_T((C_T, \mathcal{E}_T), (C'_T, \mathcal{E}'_T))$$

is representable by a quasi-compact separated S-scheme. In other words, the diagonal of $\overline{\mathcal{V}ec}_{r,d,g}$ is representable, quasi-compact and separated.

Proof. Using the same arguments above, we can restrict to the category of locally noetherian schemes. Suppose that S is an affine connected noetherian scheme. Consider the contravariant functor

$$(T \to S) \mapsto \mathrm{Isom}\left(C_T, C'_T\right).$$

This functor is represented by a scheme B (see [ACG11, pp. 47-48]). More precisely: let $Hilb_{C \times_S C'/S}$ be the Hilbert scheme which parametrizes closed subschemes of $C \times_S C'$ flat over S. B is the open subscheme of $Hilb_{C \times_S C'/S}$ with the property that a morphism $f : T \to Hilb_{C \times_S C'/S}$ factorizes through B if and only if the projections $\pi : Z_T \to C_T$ and $\pi' : Z_T \to C'_T$ are isomorphisms, where Z_T is the closed subscheme of $C \times_S C'$ represented by f. Consider the universal pair

$$\left(Z_B, \varphi := \pi' \circ \pi^{-1} : C_B \cong Z_B \cong C'_B\right).$$

Now we prove that B is quasi-projective. By construction it is enough to show that B is contained in $Hilb^{P,\mathcal{L}}_{C \times_S C'/S}$, which parametrizes closed subschemes of $C \times_S C'/S$ with Hilbert polynomial P respect to the relatively ample line bundle $\mathcal{L}$ on $C \times_S C'/S$. Let $\mathcal{L}$ (resp. $\mathcal{L}'$) be a relatively very ample line bundle on C/S (resp. C'/S). We can take $\mathcal{L} = (\det \mathcal{E})^m$ and $\mathcal{L}' = (\det \mathcal{E}')^m$ for m big enough. Then the sheaf $\mathcal{L} \boxtimes_S \mathcal{L}'$ is relatively very ample on $C \times_S C'/S$. Using the projection π we can identify Z_B and C_B. The Hilbert polynomial of Z_B with respect to the polarization $\mathcal{L} \boxtimes_S \mathcal{L}'$ is

$$P(n) = \chi((\mathcal{L} \boxtimes_S \mathcal{L}')^n) = \chi(\mathcal{L}^n \otimes \varphi^* \mathcal{L}'^n) = \deg(\mathcal{L}^n) + \deg(\mathcal{L}'^n) + 1 - g.$$

It is clearly independent from the choice of the point in B and from Z_B, proving the quasi-projectivity. In particular, B is quasi-compact and separated over S. The proposition follows from the fact that the contravariant functor

$$(T \to B) \mapsto \mathrm{Isom}_{C_T}\left(\mathcal{E}_T, \varphi^* \mathcal{E}'_T\right)$$

is representable by a quasi-compact separated scheme over B (see the proof of [LMB00, Theorem 4.6.2.1]). $\square$

Putting together Proposition 1.2.5 and Proposition 1.2.6, we get

Corollary 1.2.7. *$\overline{\mathcal{V}ec}_{r,d,g}$ is a stack for the fpqc topology.*

We now introduce a useful open cover of the stack $\overline{\mathcal{V}ec}_{r,d,g}$. We will prove that any open subset of this cover has a presentation as quotient stack of a scheme by a suitable general linear group. In particular, $\overline{\mathcal{V}ec}_{r,d,g}$ admits a smooth surjective representable morphism from a locally noetherian scheme. Putting together this fact with Proposition 1.2.6, we get that $\overline{\mathcal{V}ec}_{r,d,g}$ is an Artin stack locally of finite type.

Proposition 1.2.8. *For any scheme S and any $n \in \mathbb{Z}$, consider the subgrupoid $\overline{\mathcal{U}}_n(S)$ of $\overline{\mathcal{V}ec}_{r,d,g}(S)$ of pairs $(p : C \to S, \mathcal{E})$ such that*

(1) $R^i p_* \mathcal{E}(n) = 0$ for any $i > 0$,
(2) $\mathcal{E}(n)$ is relatively generated by global sections, i.e. the canonical morphism $p^* p_* \mathcal{E}(n) \to \mathcal{E}(n)$ is surjective, and the induced morphism $C \to Gr(p_* \mathcal{E}(n), r)$ is a closed embedding.

Then the sheaf $p_* \mathcal{E}(n)$ is flat on S and $\mathcal{E}(n)$ is cohomologically flat over S. In particular, the inclusion $\overline{\mathcal{U}}_n \hookrightarrow \overline{\mathcal{V}ec}_{r,d,g}$ makes $\overline{\mathcal{U}}_n$ into a fibered full subcategory.

Proof. We set $\mathcal{F} := \mathcal{E}(n)$. By [Wan, Proposition 4.1.3], we know that $p_* \mathcal{F}$ is flat on S and $\mathcal{F}$ is cohomologically flat over S. Consider the following cartesian diagram

$$\begin{tikzcd} C_T \arrow[r] \arrow[d, "p_T"] & C \arrow[d, "p"] \\ T \arrow[r, "\phi"] & S \end{tikzcd}$$

By *loc. cit.*, we have that $R^i p_{T*}(\mathcal{F}_T) = 0$ for any $i > 0$ and that $\mathcal{F}_T$ is relatively generated by global sections. It remains to prove that the induced T-morphism $C_T \to Gr(p_{T*}\mathcal{F}_T, r)$ is a closed embedding. This follows easily by cohomological flatness and the base change property of the Grassmannian. $\square$

Lemma 1.2.9. *The subcategories* $\{\overline{\mathcal{U}}_n\}_{n\in\mathbb{Z}}$ *form an open cover of* $\overline{\mathcal{V}ec}_{r,d,g}$.

Proof. Let S be a scheme, $(p : C \to S, \mathcal{E})$ an object of $\overline{\mathcal{V}ec}_{r,d,g}(S)$ and n an integer. We must prove that exists an open $U_n \subset S$ with the universal property that $T \to S$ factorizes through U_n if and only if $\mathcal{E}_T$ is an object of $\overline{\mathcal{U}}_n(T)$.
We can assume S affine. Lemma 1.2.4 implies that the morphism $S \to \overline{\mathcal{U}}_n$ factors through a noetherian affine scheme. So we can suppose that S is affine and noetherian. Let $\mathcal{F} := \mathcal{E}(n)$ and U_n the subset of points of S such that:

(1) $H^i(C_s, \mathcal{F}_s) = 0$ for $i > 0$,
(2) $H^0(C_s, \mathcal{F}_s) \otimes \mathcal{O}_{C_s} \to \mathcal{F}_s$ is surjective,
(3) the induced morphism in the Grassmannian $C_s \to Gr(H^0(C_s, \mathcal{F}_s), r)$ is a closed embedding.

We must prove that U_n is open and it satifies the universal property. As in the proof of [Wan, Lemma 4.1.5], consider the open subscheme $V_n \subset S$ satisfying the first two conditions above. By definition it contains U_n and it satisfies the universal property that any morphism $T \to S$ factorizes through V_n if and only if $R^i p_{T*}\mathcal{F}_T = 0$ for any $i > 0$ and $\mathcal{F}_T$ is relatively generated by global sections. By [Wan, Proposition 4.1.3], $\mathcal{F}_{V_n}$ is cohomologically flat over V_n. This implies that the fiber over a point s of the morphism

$$C_{V_n} \to Gr(p_{V_n *}\mathcal{F}_{V_n}, r)$$

is exactly $C_s \to Gr(H^0(C_s, \mathcal{F}_s), r)$. Since the property of being a closed embedding for a morphism of proper V_n-schemes is an open condition (see [Kau05, Lemma 3.13]), it follows that U_n is an open subscheme and $\mathcal{F}_{U_n} \in \overline{\mathcal{U}}_n(U_n)$.
Viceversa, suppose now that $\phi : T \to S$ is such that $\mathcal{F}_T \in \overline{\mathcal{U}}_n(T)$. The morphism factors through V_n and for any $t \in T$

$$C_t \to Gr(H^0(C_t, \mathcal{F}_t), r)$$

is a closed embedding. Since the morphism ϕ restricted to a point $t \in T$ onto is image $\phi(t)$ is fppf, by descent the morphism $C_\phi(t) \to Gr(H^0(C_{\phi(t)}, \mathcal{F}_{\phi(t)}), r)$ is a closed embedding, or in other words $\phi(t) \in U_n$.
It remains to prove that $\{\mathcal{U}_n\}$ is a covering. It is sufficient to prove that for any point s exists n such that $\mathcal{E}_s(n)$ satisfies the conditions (1), (2) and (3). By Proposition 1.1.8, the push-forward of $\mathcal{E}$ in the stabilized family is S-flat and the cohomology groups on the fibers are the same, so for any point s in S there exists n big enough such that (1) is satisfied, and by Corollary 1.1.10 the same holds for (2) and (3). □

Remark 1.2.10. As in [Wan, Remark 4.1.7] for a scheme S and a pair $(p : C \to S, \mathcal{E}) \in \overline{\mathcal{U}_n}(S)$, the direct image $p_*(\mathcal{E}(n))$ is locally free of rank $d + r(2n-1)(g-1)$. By cohomological flatness, locally on S the morphism in the Grassmannian becomes $C \hookrightarrow Gr(V_n, r) \times S$, where V_n is a k-vector space of dimension $P(n) := d + r(2n-1)(g-1)$.

We are now going to obtain a presentation of $\overline{\mathcal{U}}_n$ as a quotient stack. Consider the Hilbert scheme of closed subschemes on the Grassmannian $Gr(V_n, r)$

$$Hilb_n := Hilb_{Gr(V_n,r)}^{\mathcal{O}_{Gr(V_n,r)}(1),Q(m)}$$

with Hilbert polinomial $Q(m) = m(d+nr(2g-2))+1-g$ relative to the Plucker line bundle $\mathcal{O}_{Gr(V_n,r)}(1)$. Let $\mathscr{C}_{(n)} \hookrightarrow Gr(V_n, r) \times Hilb_n$ be the universal curve. The Grassmannian is equipped with a universal quotient $V_n \times \mathcal{O}_{Gr(V_n,r)} \to \mathcal{E}$, where $\mathcal{E}$ is the universal vector bundle. If we pull-back this morphism on the product $Gr(V_n, r) \times Hilb_n$ and we restrict to the universal curve, we obtain a surjective morphism of vector bundles $q : V_n \otimes \mathcal{O}_{\mathscr{C}_{(n)}} \to \mathscr{E}_{(n)}$. We will call $\mathscr{E}_{(n)}$ (resp. $q : V_n \otimes \mathcal{O}_{\mathscr{C}_{(n)}} \to \mathscr{E}_{(n)}$) the *universal vector bundle (resp. universal quotient) on* $\mathscr{C}_{(n)}$. Let H_n be the open subset of $Hilb_n$ consisting of points h such that:

(1) $\mathscr{C}_{(n)h}$ is semistable,
(2) $\mathscr{E}_{(n)h}$ is properly balanced,
(3) $H^i(\mathscr{C}_{(n)h}, \mathscr{E}_{(n)h}) = 0$ for $i > 0$,
(4) $H^0(q_h)$ is an isomorphism.

The restriction of the universal curve and of the universal vector bundle on H_n defines a morphism of stacks $\Theta : H_n \to \overline{\mathcal{U}}_n$. Moreover, the Hilbert scheme $Hilb_n$ is equipped with a natural action of $GL(V_n)$ and H_n is stable for this action.

Proposition 1.2.11. *The morphism of stacks*

$$\Theta : H_n \to \overline{\mathcal{U}}_n$$

is a $GL(V_n)$-bundle (in the sense of [Wan, 2.1.4]*).*

Proof. We set $GL := GL(V_n)$. First we prove that Θ is GL-invariant, i.e.

(1) the diagram

$$\begin{tikzcd} H_n \times GL \arrow[r, "m"] \arrow[d, "pr_1"] & H_n \arrow[d, "\Theta"] \\ H_n \arrow[r, "\Theta"] & \overline{\mathcal{U}}_n \end{tikzcd}$$

where pr_1 is the projection on H_n and m is the multiplication map is commutative. Equivalently, there exists a natural transformation $\rho : pr_1^*\Theta \to m^*\Theta$.

(2) ρ satifies an associativity condition (see [Wan, 2.1.4]).

In our case ρ is the identity and it is easy to see that the second condition holds. We will fix a pair $(p : C \to S, \mathcal{E}) \in \overline{\mathcal{U}}_n(S)$ and let $f : S \to \overline{\mathcal{U}}_n$ be the associated morphism. It remains to prove that morphism $f^*\Theta$ is a principal GL-bundle. More precisely, we will prove that there exists a GL-equivariant isomorphism over S

$$H_n \times_{\overline{\mathcal{U}}_n} S \cong Isom(V_n \otimes \mathcal{O}_S, p_{S*}\mathcal{E}(n)).$$

For any S-scheme T, a T-valued point of $H_n \times_{\overline{\mathcal{U}}_n} S$ corresponds to the following data:

(1) a morphism $T \to H_n$,
(2) a T-isomorphism of schemes $\psi : C_T \cong \mathscr{C}_{(n)T}$,
(3) an isomorphism of vector bundles $\psi^*\mathscr{E}_{(n)T} \cong \mathcal{E}_T(n)$.

Consider the pull-back of the universal quotient of H_n through $T \to H_n$

$$q_T : V_n \otimes \mathcal{O}_{\mathscr{C}_{(n)T}} \to \mathscr{E}_{(n)T}.$$

If we pull-back by ψ and compose with the isomorphism of (3), we obtain a surjective morphism

$$V_n \otimes \mathcal{O}_{C_T} \to \mathcal{E}_T(n).$$

We claim that the push-forward $V_n \otimes \mathcal{O}_T \to p_{T*}(\mathcal{E}_T(n)) \cong p_*(\mathcal{E}(n))_T$ is an isomorphism, or in other words it defines a T-valued point of $Isom(V_n \otimes \mathcal{O}_S, p_{S*}\mathcal{E}(n))$. As explained in Remark 1.2.10, the sheaf $p_{T*}(\mathcal{E}(n)_T)$ is a vector bundle of rank $P(n)$, so it is enough to prove the surjectivity. We can suppose that T is noetherian and by Nakayama lemma it suffices to prove the surjectivity on the fibers. On a fiber the morphism is

$$V_n \otimes \mathcal{O}_{C_t} \to H^0(\mathscr{E}_{(n)t}) \cong H^0(\mathcal{E}_t(n))$$

which is an isomorphism by the definition of H_n.
Conversely, let T be a scheme and $V_n \otimes \mathcal{O}_T \to p_*(\mathcal{E}(n))_T$ a T-isomorphism of vector bundles. By hypothesis, $\mathcal{E}_T(n)$ is relatively generated by global sections and the induced morphism in the Grassmannian is a closed embedding. Putting everything together, we obtain a surjective map

$$V_n \otimes \mathcal{O}_{C_T} \cong p_T^* p_{T*}\mathcal{E}_T(n) \to \mathcal{E}_T(n)$$

and a closed embedding $C_T \hookrightarrow Gr(V_n, r) \times T$ which defines a morphism $T \to H_n$. If we set ψ equal to the identity $C_T = \mathscr{C}_{(n)T}$, we have a unique isomorphism of vector bundles $\psi^*\mathscr{E}_{(n)T} \cong \mathcal{E}_T(n)$. Then we have obtained a T-valued point of $H_n \times_{\overline{\mathcal{U}}_n} S$. The two constructions above are inverses of each other, concluding the proof. □

Proposition 1.2.12. *The map $\Theta : H_n \to \overline{\mathcal{U}}_n$ gives an isomorphism of stacks*

$$\overline{\mathcal{U}}_n \cong [H_n/GL(V_n)]$$

Proof. This follows from [Wan, Lemma 2.1.1.]. □

From the above presentation of $\overline{\mathcal{U}}_n$ as a quotient stack, we can now prove the smoothness of $\overline{\mathcal{V}ec}_{r,d,g}$ and compute its dimension. This will conclude the proof of Theorem 1.2.2 except for the irreducibility of $\overline{\mathcal{V}ec}_{r,d,g}$ which will be proved in Lemma 1.5.2.

Corollary 1.2.13. *The scheme H_n and the stack $\overline{\mathcal{V}ec}_{r,d,g}$ are smooth of dimension respectively $P(n)^2 + (r^2+3)(g-1)$ and $(r^2+3)(g-1)$.*

Proof. We set $Gr := Gr(V_n, r)$. Arguing as in [Sch04, Proposition 3.1.3.], we see that for any k-point $h := [C \hookrightarrow Gr] \in H_n$ the co-normal sheaf $\mathcal{I}_C/\mathcal{I}_C^2$ is locally free and we have an exact sequence:

$$0 \longrightarrow \mathcal{I}_C/\mathcal{I}_C^2 \longrightarrow \Omega^1_{Gr}|_C \longrightarrow \Omega^1_C \longrightarrow 0.$$

Applying the functor $\mathrm{Hom}_{\mathcal{O}_C}(-, \mathcal{O}_C)$, we obtain the following exact sequence of vector spaces

$$0 \longrightarrow \mathrm{Hom}_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \longrightarrow H^0(C, T_{Gr}|_C) \longrightarrow \mathrm{Hom}_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C) \longrightarrow \mathrm{Ext}^1_{\mathcal{O}_C}(\Omega^1_C, \mathcal{O}_C) \longrightarrow 0$$

Now $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{I}_C/\mathcal{I}_C^2, \mathcal{O}_C)$ is the tangent space of H_n at h. We can prove that its dimension is $P(n)^2 + (r^2+3)(g-1)$ by using the sequence above as in the proof of *loc. cit.* This implies that H_n is smooth of dimension $P(n)^2 + (r^2+3)(g-1)$. The assertion for the stack $\overline{\mathcal{V}ec}_{r,d,g}$ follows immediately from Proposition 1.2.12. □

1.3. **The Schmitt compactification $\overline{U}_{r,d,g}$.** In this section we will resume how Schmitt in [Sch04], generalizing a result of Nagaraj-Seshadri in [NS99], constructs via GIT an irreducible projective variety, which is a good moduli space (for the definition see Appendix B) for an open substack of $\overline{\mathcal{V}ec}_{r,d,g}$.

First we recall the Seshadri's definition of slope-(semi)stable sheaf for a stable curve in the case of the canonical polarization.

Definition 1.3.1. Let C be a stable curve and let $C_1, \ldots, C_s$ be its irreducible components. We will say that a sheaf $\mathcal{E}$ is *P-(semi)stable* if it is torsion free of uniform rank r and for any subsheaf $\mathcal{F}$ we have

$$\frac{\chi(\mathcal{F})}{\sum s_i \omega_{C_i}} \underset{(\leq)}{<} \frac{\chi(\mathcal{E})}{r\omega_C}$$

where s_i is the rank of $\mathcal{F}$ at C_i. A P-semistable sheaf has a Jordan-Holder filtration with P-stable factors. Two P-semistable sheaves are *equivalent* if they have the same Jordan-Holder factors. Two equivalence classes are said to be *aut-equivalent* if they differ by an automorphism of the curve.

Consider the stack $\mathcal{TF}_{r,d,g}$ of torsion free sheaves of uniform rank r and Euler characteristic $d+r(1-g)$ on stable curves of genus g. Pandharipande has proved in [Pan96] that exists an open substack $\mathcal{TF}^{ss}_{r,d,g}$ which admits a projective irreducible variety as good moduli space. More precisely, this variety is a coarse moduli space for the aut-equivalence classes of P-semistable sheaves over stables curves (see [Pan96, Theorem 9.1.1]). This is the reason why we prefer the "P" instead of "slope" in the definition above.

Consider the open substack $\overline{\mathcal{V}ec}^{P(s)s}_{r,d,g} \subset \overline{\mathcal{V}ec}_{r,d,g}$ of pairs $(C, \mathcal{E})$ such that the sheaf $\pi_*\mathcal{E}$ over the stabilized curve C^{st} is P-(semi)stable. Sometimes we will simply say that the pair $(C^{st}, \pi_*\mathcal{E})$ is P-(semi)stable. As we will see in the next proposition, the set of such pairs is bounded.

Proposition 1.3.2. *The stack $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ is quasi-compact.*

Proof. By construction, it is sufficient to prove that there exists n big enough such that $\overline{\mathcal{V}ec}^{Pss}_{r,d,g} \subset \overline{\mathcal{U}}_n$. It is enough showing that there exists n big enough such that $\mathcal{E}(n)$ satisfies the conditions of Proposition 1.1.9, for any k-point $(C, \mathcal{E})$ in $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$.

Consider the set $\{(C, \mathcal{E})\}$ of k-points in $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$. Denote with $\pi : C \to C^{st}$ the stabilization morphism. Let C be a semistable curve, let $R \subset C$ be a subcurve obtained as union of some maximal chains. We set, as usual, $\widetilde{C} = R^c$ and $D := |\widetilde{C} \cap R|$. Consider the exact sequence

$$0 \longrightarrow \mathcal{I}_{D^{st}}(\pi_*\mathcal{E}) \longrightarrow \pi_*\mathcal{E} \longrightarrow (\pi_*\mathcal{E})_{D^{st}} \longrightarrow 0,$$

Observe that the cokernel is a torsion sheaf. By construction $\chi((\pi_*\mathcal{E})_{D^{st}}) = h^0((\pi_*\mathcal{E})_{D^{st}}) \leq 2rN$, where N is the number of nodes on C^{st}. A stable curve of genus g can have at most $3g-3$ nodes. By [Pan96], the set of P-semistable torsion free sheaves with $\chi = d + r(1-g)$ on stable curves of genus g is bounded. This allows us, using the theory of relative Quot schemes, to construct a quasi-compact scheme which is the fine moduli space for the pairs $(X, q : \mathcal{P} \to \mathcal{F})$ where X is a stable curve of genus g, q is a surjective morphism of sheaves on X, P is a P-semistable torsion free and $\mathcal{F}$ is a sheaf with constant Hilbert polynomial less or equal than $6r(g-1)$. In particular, up to twisting by a suitable power of the canonical bundle, we can

assume that the sheaf $\mathcal{I}_{D^{st}}(\pi_*\mathcal{E})$ is generated by global sections and that $H^1(C^{st},\mathcal{I}_{D^{st}}(\pi_*\mathcal{E}))=0$ for any k-point $(C,\mathcal{E})$ in $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ and any collection R of maximal chains in C_{exc}. By Corollary 1.1.13(ii), we have

$$\mathcal{I}_{D^{st}}(\pi_*\mathcal{E})\cong\pi_*(\mathcal{I}_D\mathcal{E}_{\widetilde{C}})=\pi_*(\mathcal{I}_R\mathcal{E}).$$

Observe that $H^i(C^{st},\pi_*(\mathcal{I}_D\mathcal{E}_{\widetilde{C}}))=H^i(\widetilde{C},\mathcal{I}_D\mathcal{E}_{\widetilde{C}})$ for $i=0,1$. In particular $\mathcal{E}$ satisfies the condition (i) of Proposition 1.1.9. Suppose that R is a maximal chain, $D=\{p,q\}$ and $D^{st}=x$. So, the fact that $H^0(C^{st},\pi_*\left(\mathcal{I}_D\mathcal{E}_{\widetilde{C}}\right))$ generates $\pi_*\left(\mathcal{I}_D\mathcal{E}_{\widetilde{C}}\right)$ implies that

$$H^0(\widetilde{C},\mathcal{I}_{p,q}\mathcal{E}_{\widetilde{C}})\to\pi_*\left(\mathcal{I}_{p,q}\mathcal{E}_{\widetilde{C}}\right)_{\{x\}}=\left(\mathcal{I}_{p,q}\mathcal{E}_{\widetilde{C}}\right)_{\{p\}}\oplus\left(\mathcal{I}_{p,q}\mathcal{E}_{\widetilde{C}}\right)_{\{q\}}$$

is surjective. In other words $\mathcal{E}$ satisfies the condition (ii) of *loc. cit.*
For the rest of the proof R will be the exceptional curve C_{exc}. Set $\mathcal{G}:=\mathcal{I}_D\mathcal{E}_{\widetilde{C}}$. Let p and q (not necessarily distinct) points on $C\backslash R=\widetilde{C}\backslash D$. Consider the exact sequence of sheaves on $\widetilde{C}$.

$$0\longrightarrow\mathcal{I}_{p,q}\mathcal{G}\longrightarrow\mathcal{G}\longrightarrow\mathcal{G}/\mathcal{I}_{p,q}\mathcal{G}\longrightarrow 0,$$

where, when $p=q$, we denote with $\mathcal{I}_{p,p}\mathcal{G}$ the sheaf $\mathcal{I}_p^2\mathcal{G}$. If we show that $H^1(\widetilde{C},\mathcal{I}_{p,q}\mathcal{G})=H^1(C^{st},\pi_*(\mathcal{I}_{p,q}\mathcal{G}))$ is zero for any k-point $(C,\mathcal{E})$ in $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ then the conditions (iii) and (iv) are satisfied for any pair in $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}(k)$. We have already shown that the pairs $(C^{st},\pi_*\mathcal{G})$ are bounded. As before the sheaf $G/\mathcal{I}_{p,q}$ is torsion and its Euler characteristic is $2r$ (not depend from the choice of p and q). Arguing as above, we can conclude that $H^1(C^{st},\pi_*(\mathcal{I}_{p,q}\mathcal{G}))=0$. □

Schmitt proves in [Sch04] that there exists an open substack $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}\subset\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ which admits a good moduli space $\overline{U}_{r,d,g}$. We recall briefly the conctruction of such space following [Sch04, pp. 174-175].
Gieseker has shown in [Gie82] that the coarse moduli space of stable curves $\overline{M}_g$ can be constructed via GIT. More precisely $\overline{M}_g\cong H_g/\!\!/_{\mathcal{L}_{H_g}}SL(W)$, where H_g is the Hilbert scheme of stable curves embedded with ω^{10} in $\mathbb{P}(W)=\mathbb{P}^{10(2g-2)-g}$, while $\mathcal{L}_{H_g}$ is a suitable $SL(W)$-linearized ample line bundle on H_g. Let $C_g\to H_g$ be the universal curve. Consider the relative Quot scheme

$$\rho:Q:=Quot(C_g/H_g,V_n\otimes\mathcal{O}_{C_g},\omega^{10}_{C_g/H_g})\to H_g.$$

We have a natural action of $SL(V_n)\times SL(W)$, linearized with respect to a suitable ρ-ample line bundle $\mathcal{L}_Q$. With an abuse of notation, we will denote again with Q the open (and closed) subscheme of Q consisting of sheaves with Euler characteristic equal to $P(n)=\dim V_n$ and uniform rank r. We set $\mathcal{L}_a:=\mathcal{L}_Q\otimes\rho^*\mathcal{L}^a_{H_g}$. For $a\gg 0$ the GIT-quotient $\overline{Q}:=Q/\!\!/_{\mathcal{L}_a}SL(W)$ exists and it is the coarse moduli space for the functor which sends a scheme S to the set of isomorphism classes of pairs $(C_S\to S,q_S:V_n\otimes\mathcal{O}_{C_S}\to\mathcal{E})$ where $C_S\to S$ is a family of stable curve and q_S is a surjective morphism of S-flat sheaves with $\chi(E_s)=P(n)$ and uniform rank r. Moreover $\overline{Q}$ is equipped with a $SL(V_n)$-linearized line bundle $\mathcal{L}_{\overline{Q}}$.
Consider now the scheme H_n defined at page 13. It has a natural $SL(V_n)$-linearized line bundle $\mathcal{L}_{Hilb}$, the semistable points for this linearized action are called *Hilbert semistable* points and their description is an open problem (see [TiB98] for some partial results in this direction). Let

$$(\mathscr{C}_{(n)},q:V_n\otimes\mathcal{O}_{\mathscr{C}_{(n)}}\to\mathscr{E}_{(n)})$$

be the universal pair on H_n. Consider the stabilized curve $\pi:\mathscr{C}_{(n)}\to\mathscr{C}^{st}_{(n)}$. The push-forward $\pi_*(q)$ (as in [Sch04, p. 180]) defines a morphism $H_n\to\overline{Q}$. The closure of the graph $\overline{\Gamma}\hookrightarrow H_n\times\overline{Q}$ gives us a $SL(V_n)$-linearized ample line bundle $\mathcal{L}^m_{Hilb}\boxtimes\mathcal{L}^a_{\overline{Q}}$. For $a\gg 0$, Schmitt has proved that the semistable points are contained in the graph (see [Sch04, Theorem 2.1.2]). Therefore, we can view such semistable points inside H_n and call them *H-semistable*.

Remark 1.3.3. An H-semistable point has the following properties (see [Sch04, Def. 2.2.10]): let, as usual, $\pi:C\to C^{st}$ be the stabilization morphism and $(C,\mathcal{E})$ is a pair in $\overline{\mathcal{U}}_n$.

(i) Suppose that C is smooth. Then $(C,\mathcal{E})$ is H-(semi)stable if and only if $(C,\mathcal{E})$ is P-(semi)stable. In this case we will say just $(C,\mathcal{E})$ is *(semi)-stable.*
(ii) We have the following chain of implications:
$(C^{st},\pi_*\mathcal{E})$ P-stable $\Rightarrow$ $(C,\mathcal{E})$ H-stable $\Rightarrow$ $(C,\mathcal{E})$ H-semistable $\Rightarrow$ $(C^{st},\pi_*\mathcal{E})$ P-semistable.

(iii) Suppose that $(C^{st}, \pi_* E)$ is strictly P-semistable. Then $(C, \mathcal{E})$ is H-semistable if and only if for every one-parameter subgroup λ of $SL(V_n)$ such that $(C^{st}, \pi_*\mathcal{E})$ is strictly P-semistable with respect to λ then $(C, \mathcal{E})$ is Hilbert-semistable with respect to λ.

A priori the H-semistability is a property of points in H_n, i.e. $[C \hookrightarrow Gr(V_n, r)]$. However it is easy to see that it depends only on the curve and the restriction of universal bundle to the curve.
In his construction Schmitt just requires that a vector bundle must be admissible, but not necessarily balanced. The next lemma proves that the vector bundles appearing in his construction are indeed also properly balanced.

Lemma 1.3.4. *If $(C^{st}, \pi_*\mathcal{E})$ is P-semistable then $\mathcal{E}$ is properly balanced.*

Proof. By considerations above, we must prove that $\mathcal{E}$ is balanced. By Remark 1.1.15(iv), we have to prove that for any connected subcurve $Z \subset C$ such that Z^c is connected, we have

$$\frac{\chi(\mathcal{F})}{\omega_Z} \leq \frac{\chi(\mathcal{E})}{\omega_C},$$

where $\mathcal{F}$ is the subsheaf of $\mathcal{E}_Z$ of sections that vanishes on $Z \cap Z^c$. Observe that $\mathcal{F}$ is also a subsheaf of $\mathcal{E}$. The hypothesis and the fact that the push-forward is left exact imply

$$\frac{\chi(\pi_*\mathcal{F})}{\omega_{Z^{st}}} \leq \frac{\chi(\pi_*\mathcal{E})}{\omega_{C^{st}}} = \frac{\chi(\mathcal{E})}{\omega_C},$$

where Z^{st} is the reduced subcurve $\pi(Z)$. It is clear that $\omega_{Z^{st}} = \omega_Z$. We have an exact sequence of vector spaces

$$0 \longrightarrow H^1\left(Z^{st}, \pi_*\mathcal{F}\right) \longrightarrow H^1\left(Z, \mathcal{F}\right) \longrightarrow H^0(Z^{st}, R^1\pi_*\mathcal{F}) \longrightarrow 0.$$

This implies $\chi(\mathcal{F}) \leq \chi(\pi_*\mathcal{F})$, concluding the proof. □

1.4. **Properties and the rigidified moduli stack $\overline{\mathcal{V}}_{r,d,g}$.** The stack $\overline{\mathcal{V}ec}_{r,d,g}$ admits a *universal curve* $\overline{\pi} : \overline{\mathcal{V}ec}_{r,d,g,1} \to \overline{\mathcal{V}ec}_{r,d,g}$, i.e. a stack $\overline{\mathcal{V}ec}_{r,d,g,1}$ and a representable morphism $\overline{\pi}$ with the property that for any morphism from a scheme S to $\overline{\mathcal{V}ec}_{r,d,g}$ associated to a pair $(C \to S, \mathcal{E})$ there exists a morphism $C \to \overline{\mathcal{V}ec}_{r,d,g,1}$ such that the diagram

$$\begin{tikzcd} C \arrow[r] \arrow[d] & \overline{\mathcal{V}ec}_{r,d,g,1} \arrow[d, "\overline{\pi}"] \\ S \arrow[r] & \overline{\mathcal{V}ec}_{r,d,g} \end{tikzcd}$$

is cartesian. Furthermore, the universal curve admits a *universal vector bundle*, i.e. for any morphism from a scheme S to $\overline{\mathcal{V}ec}_{r,d,g}$ associated to a pair $(C \to S, \mathcal{E})$, we associate the vector bundle $\mathcal{E}$ on C. This allows us to define a coherent sheaf for the site lisse-étale on $\overline{\mathcal{V}ec}_{r,d,g,1}$ flat over $\overline{\mathcal{V}ec}_{r,d,g}$. The stabilization morphism induces a morphism of stacks

$$\overline{\phi}_{r,d} : \overline{\mathcal{V}ec}_{r,d,g} \longrightarrow \overline{\mathcal{M}}_g$$

which forgets the vector bundle and sends the curve in its stabilization. We will denote with $\mathcal{V}ec_{r,d,g}$ (resp. $\mathcal{U}_n$) the open substack of $\overline{\mathcal{V}ec}_{r,d,g}$ (resp. $\overline{\mathcal{U}}_n$) of pairs $(C, \mathcal{E})$ where C is a smooth curve. In the next sections we will often need the restriction of $\overline{\phi}_{r,d}$ to the open locus of smooth curves

$$\phi_{r,d} : \mathcal{V}ec_{r,d,g} \longrightarrow \mathcal{M}_g.$$

The group $\mathbb{G}_m$ is contained in a natural way in the automorphism group of any object of $\overline{\mathcal{V}ec}_{r,d,g}$, as multiplication by scalars on the vector bundle. There exists a procedure for removing these automorphisms, called $\mathbb{G}_m$*-rigidification* (see [ACV03, Section 5]). We obtain an irreducible smooth Artin stack $\overline{\mathcal{V}}_{r,d,g} := \overline{\mathcal{V}ec}_{r,d,g} \,/\!\!/\mathbb{G}_m$ of dimension $(r^2+3)(g-1)+1$, with a surjective smooth morphism $\nu_{r,,d} : \overline{\mathcal{V}ec}_{r,d,g} \to \overline{\mathcal{V}}_{r,d,g}$. The forgetful morphism $\overline{\phi}_{r,d} : \overline{\mathcal{V}ec}_{r,d,g} \longrightarrow \overline{\mathcal{M}}_g$ factorizes through the forgetful morphism $\overline{\phi}_{r,d} : \overline{\mathcal{V}}_{r,d,g} \to \overline{\mathcal{M}}_g$.

Over the locus of smooth curves we have the following diagram

$$\begin{tikzcd}
\mathcal{V}ec_{r,d,g} \arrow[rr, "\nu_{r,d}"] \arrow[d, "det"] & & \mathcal{V}_{r,d,g} \arrow[d, "\widetilde{det}"] \\
\mathcal{J}ac_{d,g} \arrow[rr, "\nu_{1,d}"] \arrow[dr] & & \mathcal{J}_{d,g} \arrow[dl] \\
& \mathcal{M}_g &
\end{tikzcd}$$

where det (resp. $\widetilde{det}$) is the determinant morphism, which send an object $(C \to S, \mathcal{E}) \in \mathcal{V}ec_{r,d,g}(S)$ (resp. $\in \mathcal{V}_{r,d,g}(S)$) to $(C \to S, \det \mathcal{E}) \in \mathcal{J}ac_{d,g}(S)$ (resp. $\in \mathcal{J}_{d,g}(S)$). Observe that the obvious extension on $\overline{\mathcal{V}ec}_{r,d,g}$ of the determinant morphism does not map to the compactified universal Jacobian $\overline{\mathcal{J}ac}_{d,g}$, because the basic inequalities for $\overline{\mathcal{J}ac}_{d,g}$ are more restrictive.

1.5. **Local structure.** The local structure of the stack $\overline{\mathcal{V}ec}_{r,d,g}$ is governed by the deformation theory of pairs $(C, \mathcal{E})$, where C is a semistable curve and $\mathcal{E}$ is a properly balanced vector bundle. Therefore we are going to review the necessary facts. First of all, the deformation functor Def_C of a semistable curve C is smooth (see [Ser06, Proposition 2.2.10(i), Proposition 2.4.8]) and it admits a miniversal deformation ring (see [Ser06, Theorem 2.4.1]), i.e. there exists a formally smooth morphism of functors of local Artin k-algebras

$$\mathrm{Spf}\, k[\![x_1, \ldots, x_N]\!] \to \mathrm{Def}_C, \text{ where } N := \mathrm{ext}^1(\Omega_C, \mathcal{O}_C)$$

inducing an isomorphism between the tangent spaces. Moreover, if C is stable its deformation functor admits a universal deformation ring (see [Ser06, Corollary 2.6.4]), i.e. the morphism of functors above is an isomorphism. Let x be a singular point of C and $\hat{\mathcal{O}}_{C,x}$ the completed local ring of C at x. The deformation functor $\mathrm{Def}_{\mathrm{Spec}\hat{\mathcal{O}}_{C,x}}$ admits a miniversal deformation ring $k[\![t]\!]$ (see [DM69, pag. 81]). Let Σ be the set of singular points of C. The morphism of Artin functors

$$loc : \mathrm{Def}_C \to \prod_{x \in \Sigma} \mathrm{Def}_{\mathrm{Spec}\hat{\mathcal{O}}_{C,x}}$$

is formally smooth (see [DM69, Proposition 1.5]). For a vector bundle $\mathcal{E}$ over C, we will denote with $\mathrm{Def}_{(C,\mathcal{E})}$ the deformation functor of the pair (for a more precise definition see [CMKV15, Def. 3.1]). As in [CMKV15, Def. 3.4], the automorphism group $\mathrm{Aut}(C, \mathcal{E})$ (resp. $\mathrm{Aut}(C)$) acts on $\mathrm{Def}_{(C,\mathcal{E})}$ (resp. Def_C). Using the same argument of [CMKV15, Lemma 5.2], we can see that the multiplication by scalars on $\mathcal{E}$ acts trivially on $\mathrm{Def}_{(C,\mathcal{E})}$. By [FGI$^+$05, Theorem 8.5.3], the forgetful morphism

$$\mathrm{Def}_{(C,\mathcal{E})} \to \mathrm{Def}_C$$

is formally smooth and the tangent space of $\mathrm{Def}_{(C,\mathcal{E})}$ has dimension $\mathrm{ext}^1(\Omega_C, \mathcal{O}_C) + \mathrm{ext}^1(\mathcal{E}, \mathcal{E})$. Let $h := [C \hookrightarrow Gr(V_n, r)]$ be a k-point of H_n. Let $\hat{\mathcal{O}}_{H_n,h}$ be the completed local ring of H_n at h. Clearly, the ring $\hat{\mathcal{O}}_{H_n,h}$ is a universal deformation ring for the deformation functor Def_h of the closed embedding h. Moreover

Lemma 1.5.1. *The natural morphism*

$$Def_h \to Def_{(C,\mathcal{E})}$$

is formally smooth.

Proof. For any k-algebra R, we will set $Gr(V_n, r)_R := Gr(V_n, r) \times_k \mathrm{Spec}R$. We have to prove that given

(1) a surjection $B \to A$ of Artin local k-algebras,
(2) a deformation $h_A := [C_A \hookrightarrow Gr(V_n, r)_A]$ of h over A
(3) a deformation $(C_B, \mathcal{E}_B)$ of $(C, \mathcal{E})$ over B, which is a lifting of $(C_A, \mathcal{E}_A)$,

then there exists an extension h_B over B of h_A which maps on $(C_B, \mathcal{E}_B)$. Since by hypothesis $H^1(C, \mathcal{E}(n)) = 0$, we can show that the restriction map $res : H^0(C_B, \mathcal{E}_B(n)) \to H^0(C_A, \mathcal{E}_A(n))$ is surjective. Now h_A only depends on the vector bundle $\mathcal{E}_A$ and on the choiche of a basis for $H^0(C_A, \mathcal{E}_A(n))$. We can lift the basis, using the map res, to a basis $\mathcal{B}$ of $H^0(C_B, \mathcal{E}_B(n))$. The basis $\mathcal{B}$ induces a morphism $C_B \to Gr(V_n, r)_B$ which is a lifting for h_A. □

The next lemma concludes the proof of Theorem 1.2.2.

Proposition 1.5.2. *The stack $\overline{\mathcal{V}ec}_{r,d,g}$ is irreducible.*

Proof. Since the morphism $loc : \mathrm{Def}_C \to \prod_{x\in\Sigma} \mathrm{Def}_{\mathrm{Spec}\hat{\mathcal{O}}_{C,x}}$ is formally smooth, Lemma 1.5.1 implies that the morphism $Def_h \to \prod_{x\in\Sigma} \mathrm{Def}_{\mathrm{Spec}\hat{\mathcal{O}}_{C,x}}$ is formally smooth. In particular, any semistable curve with a properly balanced vector bundle can be deformed to a smooth curve with a vector bundle. In other words, the open substack $\mathcal{V}ec_{r,d,g}$ is dense in $\overline{\mathcal{V}ec}_{r,d,g}$; hence $\overline{\mathcal{V}ec}_{r,d,g}$ is irreducible if and only if $\mathcal{V}ec_{r,d,g}$ is irreducible. And this follows from the fact that $\mathcal{M}_g$ is irreducible and that the morphism $\mathcal{V}ec_{r,d,g} \to \mathcal{M}_g$ is open (because is flat and locally of finite presentation) with irreducible geometric fibers (by [Hof10, Corollary A.5]). □

We are now going to construct a miniversal deformation ring for $\mathrm{Def}_{(C,\mathcal{E})}$ by taking a slice of H_n.

Lemma 1.5.3. *Let $h := [C \hookrightarrow Gr(V_n, r)]$ a k-point of H_n and let $\mathcal{E}$ be the restriction to C of the universal vector bundle. Assume that $Aut(C,\mathcal{E})$ is smooth and linearly reductive. Then the following hold.*

(i) There exists a slice for H_n. More precisely, there exists a locally closed $Aut(C,\mathcal{E})$-invariant subset U of H_n, with $h \in U$, such that the natural morphism

$$U \times_{Aut(C,\mathcal{E})} GL(V_n) \to H_n$$

is étale and affine and moreover the induced morphism of stacks

$$[U/Aut(C,\mathcal{E})] \to \overline{\mathcal{U}}_n$$

is affine and étale.

(ii) The completed local ring $\hat{\mathcal{O}}_{U,h}$ of U at h is a miniversal deformation ring for $Def_{(C,\mathcal{E})}$.

Proof. The part (i) follows from [Alp10, Theorem 3]. We will prove the second one following the strategy of [CMKV15, Lemma 6.4]. We will set $F \subset Def_h$ as the functor pro-represented by $\hat{\mathcal{O}}_{U,h}$, $G := GL(V_n)$ and $N := \mathrm{Aut}(C,\mathcal{E})$. Since $\mathrm{Def}_h \to \mathrm{Def}_{(C,\mathcal{E})}$ is formally smooth, it is enough to prove that the restriction to $F(A)$ of $\mathrm{Def}_h(A) \to \mathrm{Def}_{(C,\mathcal{E})}(A)$ is surjective for any local Artin k-algebra A and bijective when $A = k[\epsilon]$. Let $\mathfrak{g}$ (resp. $\mathfrak{n}$) be the deformation functor pro-represented by the completed local ring of G (resp. N) at the identity. There is a natural map $\mathfrak{g}/\mathfrak{n} \to \mathrm{Def}_h$ given by the derivative of the orbit map. More precisely, for a local Artin k-algebra A:

$$\begin{array}{ccc} \mathfrak{g}/\mathfrak{n}(A) & \to & \mathrm{Def}_h(A) \\ [g] & \mapsto & g.v^{triv} \end{array}$$

where v^{triv} is the trivial deformation over $\mathrm{Spec}A$. First of all we will construct a morphism $\mathrm{Def}_h \to \mathfrak{g}/\mathfrak{n}$ such that the derivative of the orbit map defines a section. The construction is the following: up to étale base change, the morphism $U \times_N G \to H_n$ of part (i), admits a section locally on h. The morphism, obtained composing this section with the morphism $U \times_N G \to G/N$, which sends a class $[(u,g)]$ to $[g]$, induces a morphism of Artin functors

$$\mathrm{Def}_h \to \mathfrak{g}/\mathfrak{h}$$

with the desired property. By construction, if A is a local Artin k-algebra then the inverse image of $0 \in \mathfrak{g}/\mathfrak{n}(A)$ is $F(A)$. If $v \in \mathrm{Def}_h(A)$ maps to some element $[g] \in \mathfrak{g}/\mathfrak{n}(A)$ then $g^{-1}v \in F(A)$. Because both v and $g^{-1}v$ map to the same element of $\mathrm{Def}_{(C,\mathcal{E})}$, we can conclude that $F(A) \to \mathrm{Def}_{(C,\mathcal{E})}(A)$ is surjective.
It remains to prove the injectivity of $F(k[\epsilon]) \to \mathrm{Def}_{(C,\mathcal{E})}(k[\epsilon])$. We consider the following complex of k-vector spaces

$$0 \to \mathfrak{g}/\mathfrak{n} \to Def_h(k[\epsilon]) \to Def_{(C,\mathcal{E})}(k[\epsilon]) \to 0$$

where the first map is the derivative of the orbit map. We claim that this is an exact sequence, which would prove the injectivity of $F(k[\epsilon]) \to \mathrm{Def}_{(C,\mathcal{E})}(k[\epsilon])$ by the definition of F. The only non obvious thing to check is the exactness in the middle. Suppose that $h_{k[\epsilon]} \in \mathrm{Def}_h(k[\epsilon])$ is trivial in $\mathrm{Def}_{(C,\mathcal{E})}(k[\epsilon])$, i.e. if $q_\epsilon : V_n \otimes \mathcal{O}_{C_\epsilon} \to \mathcal{E}_\epsilon$ represents the embedding $h_{k[\epsilon]}$, then there exists an isomorphism with the trivial deformation on $k[\epsilon]$: $\varphi : C_\epsilon \cong C[\epsilon]$ and $\psi : \varphi_*\mathcal{E}_\epsilon \cong \mathcal{E}[\epsilon]$. Consider the morphism

$$g_\epsilon := \psi \circ \varphi_* q_\epsilon : V_n \otimes \mathcal{O}_{C[\epsilon]} \to \mathcal{E}[\epsilon]$$

which represents the same class $h_{k[\epsilon]}$. By definition of H_n, the push-forward of g_ϵ on $k[\epsilon]$ is an isomorphism

$$V_n \otimes k[\epsilon] \to H^0(C, \mathcal{E}(n)) \otimes k[\epsilon]$$

and it defines uniquely the class $h_{k[\epsilon]}$. We can choose basis for V_n and $H^0(C,\mathcal{E}(n))$ such that g_ϵ differs from the trivial deformation of $\mathrm{Def}_h(k[\epsilon])$ by an invertible matrix $g \equiv Id \bmod \epsilon$, which concludes the proof. □

2. Preliminaries about line bundles on stacks.

2.1. **Picard group and Chow groups of a stack.** We will recall the definitions and some properties of the Picard group and the Chow group of an Artin stack. Some parts contains overlaps with [MV14, Section 2.9]. Let $\mathcal{X}$ be an Artin stack locally of finite type over k.

Definition 2.1.1. [Mum65, p.64] A *line bundle* $\mathcal{L}$ on $\mathcal{X}$ is the data consisting of a line bundle $\mathcal{L}(F_S) \in \mathrm{Pic}(S)$ for every scheme S and morphism $F_S : S \to \mathcal{X}$ such that:

- For any commutative diagram
$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "F_S"'] & & T \arrow[dl, "F_T"] \\ & \mathcal{X} & \end{tikzcd}$$
there is an isomorphism $\phi(f) : \mathcal{L}(F_S) \cong f^*\mathcal{L}(F_T)$.
- For any commutative diagram
$$\begin{tikzcd} S \arrow[r, "f"] \arrow[dr, "F_S"'] & T \arrow[r, "g"] \arrow[d, "F_T"] & Z \arrow[dl, "F_Z"] \\ & \mathcal{X} & \end{tikzcd}$$
we have the following commutative diagram of isomorphisms
$$\begin{tikzcd} \mathcal{L}(F_S) \arrow[r, "\phi(f)"] \arrow[d, "\phi(g\circ f)"] & f^*\mathcal{L}(F_T) \arrow[d, "f^*\phi(g)"] \\ (g\circ f)^*\mathcal{L}(F_Z) \arrow[r, "\cong"] & f^*g^*\mathcal{L}(F_Z) \end{tikzcd}$$

The abelian group of isomorphism classes of line bundles on $\mathcal{X}$ is called the *Picard group* of $\mathcal{X}$ and is denoted by $\mathrm{Pic}(\mathcal{X})$.

Remark 2.1.2. The definition above is equivalent to have a locally free sheaf of rank 1 for the site lisse-étale ([Bro, Proposition 1.1.1.4.]).

If $\mathcal{X}$ is a quotient stack $[X/G]$, where X is a scheme of finite type over k and G a group scheme of finite type over k, then $\mathrm{Pic}(\mathcal{X}) \cong \mathrm{Pic}(X)^G$ (see [ACG11, Chap. XIII, Corollary 2.20]), where $\mathrm{Pic}(X)^G$ is the group of isomorphism classes of G-linearized line bundles on X.

In [EG98, Section 5.3] (see also [Edi13, Definition 3.5]) Edidin and Graham introduce the operational Chow groups of an Artin stack $\mathcal{X}$, as generalization of the operational Chow groups of a scheme.

Definition 2.1.3. A *Chow cohomology class* c on $\mathcal{X}$ is the data consisting of an element $c(F_S)$ in the operational Chow group $A^*(S) = \oplus A^i(S)$ for every scheme S and morphism $F_S : S \to \mathcal{X}$ such that for any commutative diagram
$$\begin{tikzcd} S \arrow[rr, "f"] \arrow[dr, "F_S"'] & & T \arrow[dl, "F_T"] \\ & \mathcal{X} & \end{tikzcd}$$
we have $c(F_S) \cong f^*c(F_T)$, with the obvious compatibility requirements. The abelian group consisting of all the i-th Chow cohomology classes on $\mathcal{X}$ together with the operation of sum is called the *i-th Chow group* of $\mathcal{X}$ and is denoted by $A^i(\mathcal{X})$.

If $\mathcal{X}$ is a quotient stack $[X/G]$, where X is a scheme of finite type over k and G a group scheme of finite type over k, then $A^i(\mathcal{X}) \cong A^i_G(X)$ (see [EG98, Proposition 19]), where $A^i_G(X)$ is the operational equivariant Chow group defined in [EG98, Section 2.6]. We have a homomorphism of groups $c_1 : \mathrm{Pic}(\mathcal{X}) \to A^1(\mathcal{X})$ defined by the first Chern class.

The next theorem resumes some results on the Picard group of a smooth stack, which will be useful for our purposes.

Theorem 2.1.4. *Let $\mathcal{X}$ be a (not necessarily quasi-compact) smooth Artin stack over k. Let $\mathcal{U} \subset \mathcal{X}$ be an open substack.*

(i) The restriction map $Pic(\mathcal{X}) \to Pic(\mathcal{U})$ is surjective.
(ii) If $\mathcal{X} \backslash \mathcal{U}$ has codimension ≥ 2 in $\mathcal{X}$, then $Pic(\mathcal{X}) = Pic(\mathcal{U})$.

Suppose that $\mathcal{X} = [X/G]$ where G is an algebraic group and X is a smooth quasi-projective variety with a G-linearized action

(iii) The first Chern class map $c_1 : Pic(\mathcal{X}) \to A^1(\mathcal{X})$ is an isomorphism.
(iv) If $\mathcal{X} \backslash \mathcal{U}$ has codimension 1 with irreducible components $\mathcal{D}_i$, then we have an exact sequence

$$\bigoplus_i \mathbb{Z}\langle \mathcal{O}_{\mathcal{X}}(\mathcal{D}_i)\rangle \longrightarrow Pic(\mathcal{X}) \longrightarrow Pic(\mathcal{U}) \longrightarrow 0$$

Proof. The first two points are proved in [BH12, Lemma 7.3].The third point follows from [EG98, Corollary 1]. The last one follows from [EG98, Proposition 5] $\square$

2.2. **Determinant of cohomology and Deligne pairing.** There exists two methods to produce line bundles on a stack parametrizing nodal curves with some extra-structure (as our stacks): the determinant of cohomology and the Deligne pairing. We will recall the main properties of these construction, following the presentation given in [ACG11, Chap. XIII, Sections 4 and 5] and the resume in [MV14, Section 2.13]. Let $p : C \to S$ be a family of nodal curves. Given a coherent sheaf $\mathcal{F}$ on C flat over S, the *determinant of cohomology* of $\mathcal{F}$ is a line bundle $d_p(\mathcal{F}) \in \mathrm{Pic}(S)$ defined as it follows. Locally on S (by Proposition B.4), there exists a complex of vector bundles $f : V_0 \to V_1$ such that $\ker f = p_*(\mathcal{F})$ and $\mathrm{coker} f = R^1p_*(\mathcal{F})$ and then we set

$$d_p(\mathcal{F}) := \det V_0 \otimes (\det V_1)^{-1}.$$

This definition does not depend on the choice of the complex $V_0 \to V_1$; in particular this defines a line bundle globally on S. The proof of the next theorem can be found in [ACG11, Chap. XIII, Section 4].

Theorem 2.2.1. *Let $p : C \to S$ be a family of nodal curves and let $\mathcal{F}$ be a coherent sheaf on C flat on S.*

(i) The first Chern class of $d_p(\mathcal{F})$ is equal to

$$c_1(d_p(\mathcal{F})) = c_1(p_!(\mathcal{F})) := c_1(p_*(\mathcal{F})) - c_1(R^1p_*(\mathcal{F})).$$

(ii) Given a cartesian diagram

$$\begin{tikzcd} C \times_S T \arrow[r, "g"] \arrow[d, "q"'] & C \arrow[d, "p"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

we have a canonical isomorphism

$$f^*d_p(\mathcal{F}) \cong d_q(g^*\mathcal{F}).$$

Given two line bundles $\mathcal{M}$ and $\mathcal{L}$ over a family of nodal curves $p : C \to S$, the *Deligne pairing* of $\mathcal{M}$ and $\mathcal{L}$ is a line bundle $\langle \mathcal{M}, \mathcal{L}\rangle_p \in Pic(S)$ which can be defined as

$$\langle \mathcal{M}, \mathcal{L}\rangle_p := d_p(\mathcal{M} \otimes \mathcal{L}) \otimes d_p(\mathcal{M})^{-1} \otimes d_p(\mathcal{L})^{-1} \otimes d_p(\mathcal{O}_C).$$

The proof of the next theorem can be found in [ACG11, Chap. XIII, Section 5].

Theorem 2.2.2. *Let $p : C \to S$ be a family of nodal curves.*

(i) The first Chern class of $\langle \mathcal{M}, \mathcal{L}\rangle_p$ is equal to

$$c_1(\langle \mathcal{M}, \mathcal{L}\rangle_p) = p_*(c_1(\mathcal{M}) \cdot c_1(\mathcal{L})).$$

(ii) Given a Cartesian diagram

$$\begin{tikzcd} C \times_S T \arrow[r, "g"] \arrow[d, "q"'] & C \arrow[d, "p"] \\ T \arrow[r, "f"] & S \end{tikzcd}$$

we have a canonical isomorphism

$$f^*\langle \mathcal{M}, \mathcal{L}\rangle_p \cong \langle g^*\mathcal{M}, g^*\mathcal{L}\rangle_q$$

Remark 2.2.3. By the functoriality of the determinant of cohomology and of the Deligne pairing, we can extend their definitions to the case when we have a representable, proper and flat morphism of Artin stacks such that the geometric fibers are nodal curves.

2.3. **Picard group of $\overline{\mathcal{M}}_g$.** The universal family $\overline{\pi}:\overline{\mathcal{M}}_{g,1}\to\overline{\mathcal{M}}_g$ is a representable, proper, flat morphism with stable curves as geometric fibers. In particular we can define the relative dualizing sheaf $\omega_{\overline{\pi}}$ on $\overline{\mathcal{M}}_{g,1}$ and taking the determinant of cohomology $d_{\overline{\pi}}(\omega_{\overline{\pi}}^n)$ we obtain line bundles on $\overline{\mathcal{M}}_g$. The line bundle $\Lambda:=d_{\overline{\pi}}(\omega_{\overline{\pi}})$ is called the *Hodge line bundle*.

Let C be a stable curve and for every node x of C, consider the partial normalization C' at x. If C' is connected then we say x node of type 0, if C' is the union of two connected curves of genus i and $g-i$, with $i\leq g-i$ (for some i), then we say that x is a node of type i. The boundary $\overline{\mathcal{M}}_g/\mathcal{M}_g$ decomposes as union of irreducible divisors δ_i for $i=0,\dots,\lfloor g/2\rfloor$, where δ_i parametrizes (as stack) the stable curves with a node of type i. The generic point of δ_0 is an irreducible curve of genus g with exactly one node, the generic point of δ_i for $i=1,\dots,\lfloor g/2\rfloor$ is a stable curve formed by two irreducible smooth curves of genus i and $g-i$ meeting in exactly one point. We set $\delta:=\sum\delta_i$. By Theorem 2.1.4 we can associate to any δ_i a unique (up to isomorphism) line bundle $\mathcal{O}(\delta_i)$. We set $\mathcal{O}(\delta)=\bigotimes_i\mathcal{O}(\delta_i)$.

The proof of the next results for $g\geq 3$ can be found in [AC87, Theorem. 1] based upon a result of [Har83]. If $g=2$ see [**?**] for $\mathrm{Pic}(\mathcal{M}_2)$ and [Cor07, Proposition 1] for $\mathrm{Pic}(\overline{\mathcal{M}}_2)$.

Theorem 2.3.1. *Assume $g\geq 2$. Then*

(i) $Pic(\mathcal{M}_g)$ is freely generated by the Hodge line bundle, except for $g=2$ in which case we add the relation $\Lambda^{10}=\mathcal{O}_{\mathcal{M}_2}$.

(ii) $Pic(\overline{\mathcal{M}}_g)$ is freely generated by the Hodge line bundle and the boundary divisors, except for $g=2$ in which case we add the relation $\Lambda^{10}=\mathcal{O}(\delta_0+2\delta_1)$.

2.4. **Picard Group of $\mathcal{J}ac_{d,g}$.** The universal family $\pi:\mathcal{J}ac_{d,g,1}\to\mathcal{J}ac_{d,g}$ is a representable, proper, flat morphism with smooth curves as geometric fibers. In particular, we can define the relative dualizing sheaf ω_π and the universal line bundle $\mathcal{L}$ on $\mathcal{J}ac_{d,g,1}$. Taking the determinant of cohomology $\Lambda(n,m):=d_\pi(\omega_\pi^n\otimes\mathcal{L}^m)$, we obtain several line bundles on $\mathcal{J}ac_{d,g}$.

The proof of next theorem can be found in [MV14, Theorem A(i) and Notation 1.5], based upon a result of [Kou91].

Theorem 2.4.1. *Assume $g\geq 2$. Then $Pic(\mathcal{J}ac_{d,g})$ is freely generated by $\Lambda(1,0)$, $\Lambda(1,1)$ and $\Lambda(0,1)$, except in the case $g=2$ in which case we add the relation $\Lambda(1,0)^{10}=\mathcal{O}_{\mathcal{J}ac_{d,g}}$.*

2.5. **Picard Groups of the fibers.** Fix now a smooth curve C with a line bundle $\mathcal{L}$. Let $\mathcal{V}ec_{=\mathcal{L},C}$ be the stack whose objects over a scheme S are the pairs $(\mathcal{E},\varphi)$ where $\mathcal{E}$ is a vector bundle of rank r on $C\times S$ and φ is an isomorphism between the line bundles $\det\mathcal{E}$ and $\mathcal{L}\boxtimes\mathcal{O}_S$. A morphism between two objects over S is an isomorphism of vector bundles compatible with the isomorphism of determinants. $\mathcal{V}ec_{=\mathcal{L},C}$ is a smooth Artin stack of dimension $(r^2-1)(g-1)$. We denote with $\mathcal{V}ec^{(s)s}_{=\mathcal{L},C}$ the open substack of (semi)stable vector bundles. Since the set of isomorphism classes of semistable vector bundles on C is bounded, the stack $\mathcal{V}ec^{ss}_{=\mathcal{L},C}$ is quasi-compact. Consider the set of equivalence classes (defined as in Section 1.3) of semistable vector bundles over the curve C with determinant isomorphic to $\mathcal{L}$. There exists a normal projective variety $U_{\mathcal{L},C}$ which is a coarse moduli space for this set. Observe the stack $\mathcal{V}ec_{=\mathcal{L},C}$ is the fiber of the determinant morphism $det:\mathcal{V}ec_{r,d,g}\to\mathcal{J}ac_{d,g}$ with respect to the k-point $(C,\mathcal{L})$.

Theorem 2.5.1. *Let C be a smooth curve with a line bundle $\mathcal{L}$. Let $\mathcal{E}$ be the universal vector bundle over $\pi:\mathcal{V}ec_{=\mathcal{L},C}\times C\to\mathcal{V}ec_{=\mathcal{L},C}$ of rank r and degree d. Then:*

(i) We have natural isomorphisms induced by the restriction

$$\langle(d_\pi(\mathcal{E})\rangle\cong Pic(\mathcal{V}ec_{=\mathcal{L},C})\cong Pic(\mathcal{V}ec^{ss}_{=\mathcal{L},C}).$$

(ii) $U_{\mathcal{L},C}$ is a good moduli space for $\mathcal{V}ec^{ss}_{=\mathcal{L},C}$.

(iii) The good moduli morphism $\mathcal{V}ec^{ss}_{=\mathcal{L},C}\to U_{\mathcal{L},C}$ induces an exact sequence of groups

$$0\to Pic(U_{\mathcal{L},C})\to Pic(\mathcal{V}ec^{ss}_{=\mathcal{L},C})\to\mathbb{Z}/\tfrac{r}{n_{r,d}}\mathbb{Z}\to 0$$

where the second map sends $d_\pi(\mathcal{E})^k$ to k.

Proof. Part (i) is proved in [Hof12, Theorem 3.1 and Corollary 3.2]. Part (ii) follows from [Hof12, Section 2]. Part (iii) is proved in [Hof12, Theorem 3.7]. □

Remark 2.5.2. By [Hof12, Corollary 3.8], the variety $U_{\mathcal{L},C}$ is locally factorial. Moreover, except the cases when $g = r = 2$ and $\deg \mathcal{L}$ is even, the closed locus of strictly semistable vector bundles is not a divisor. So, by Theorem 2.1.4, when $(r, g, d) \neq (2, 2, 0) \in \mathbb{Z} \times \mathbb{Z} \times (\mathbb{Z}/2\mathbb{Z})$ we have that $\mathrm{Pic}(\mathcal{V}ec_{=\mathcal{L},C}) \cong \mathrm{Pic}(\mathcal{V}ec^s_{=\mathcal{L},C})$ and, since $U_{\mathcal{L},C}$ is locally factorial, $\mathrm{Pic}(U_{\mathcal{L},C}) \cong \mathrm{Pic}(U^s_{\mathcal{L},C})$.

2.6. **Boundary divisors.** The aim of this section is to study the boundary divisors of $\overline{\mathcal{V}ec}_{r,d,g}$. We first introduce some divisors contained in the boundary of $\overline{\mathcal{V}ec}_{r,d,g}$.

Definition 2.6.1. The *boundary divisors* of $\overline{\mathcal{V}ec}_{r,d,g}$ are:

- $\widetilde{\delta}_0 := \widetilde{\delta}_0^0$ is the divisor whose generic point is an irreducible curve C with just one node and $\mathcal{E}$ is a vector bundle of degree d,
- if $k_{r,d,g}|2i-1$ and $0 < i < g/2$:
  $\widetilde{\delta}_i^j$ for $0 \leq j \leq r$ is the divisor whose generic point is a curve C composed by two irreducible smooth curves C_1 and C_2 of genus i and $g-i$ meeting in one point and $\mathcal{E}$ a vector bundle over C with multidegree
  $$(\deg \mathcal{E}_{C_1}, \deg \mathcal{E}_{C_2}) = \left(d\frac{2i-1}{2g-2} - \frac{r}{2} + j, d\frac{2(g-i)-1}{2g-2} + \frac{r}{2} - j\right),$$
- if $k_{r,d,g} \nmid 2i-1$ and $0 < i < g/2$:
  $\widetilde{\delta}_i^j$ for $0 \leq j \leq r-1$ is the divisor whose generic point is a curve C composed by two irreducible smooth curves C_1 and C_2 of genus i and $g-i$ meeting in one point and $\mathcal{E}$ a vector bundle over C with multidegree
  $$(\deg \mathcal{E}_{C_1}, \deg \mathcal{E}_{C_2}) = \left(\left\lceil d\frac{2i-1}{2g-2} - \frac{r}{2}\right\rceil + j, \left\lfloor d\frac{2(g-i)-1}{2g-2} + \frac{r}{2}\right\rfloor - j\right),$$
- if g is even:
  $\widetilde{\delta}_{\frac{g}{2}}^j$ for $0 \leq j \leq \lfloor \frac{r}{2} \rfloor$ is the divisor whose generic point is a curve C composed by two irreducible smooth curves C_1 and C_2 of genus $g/2$ meeting in one point and $\mathcal{E}$ a vector bundle over C with multidegree
  $$(\deg \mathcal{E}_{C_1}, \deg \mathcal{E}_{C_2}) = \left(\left\lceil \frac{d-r}{2}\right\rceil + j, \left\lfloor \frac{d+r}{2}\right\rfloor - j\right).$$

If $i < g/2$ and $k_{r,d,g}|2i-1$ (resp. g and $d+r$ even) we will call $\widetilde{\delta}_i^0$ and $\widetilde{\delta}_i^r$ (resp. $\widetilde{\delta}_{\frac{g}{2}}^0$) the *extremal boundary divisors*. We will call *non-extremal boundary divisors* the boundary divisors which are not extremal.
By Theorem 2.1.4, we can associate to $\widetilde{\delta}_i^j$ a line bundle on $\overline{\mathcal{U}}_n$ for any n, which glue to a line bundle $\mathcal{O}(\widetilde{\delta}_i^j)$ on $\overline{\mathcal{V}ec}_{r,d,g}$, we will call them *boundary line bundles*. Moreover, if $\widetilde{\delta}_i^j$ is a (non)-extremal divisor, we will call $\mathcal{O}(\widetilde{\delta}_i^j)$ *(non)-extremal boundary line bundle.*

Indeed, it turns out that the boundary of $\overline{\mathcal{V}ec}_{r,d,g}$ is the union of the above boundary divisors.

Proposition 2.6.2.

(i) The boundary $\widetilde{\delta} := \overline{\mathcal{V}ec}_{r,d,g}/\mathcal{V}ec_{r,d,g}$ of $\overline{\mathcal{V}ec}_{r,d,g}$ is a normal crossing divisor and its irreducible components are $\widetilde{\delta}_i^j$ for $0 \leq i \leq g/2$ and $j \in J_i$ where
$$J_i = \begin{cases} 0 & \text{if } i = 0, \\ \{0, \dots, r\} & \text{if } k_{r,d,g}|2i-1 \text{ and } 0 < i < g/2, \\ \{0, \dots, r-1\} & \text{if } k_{r,d,g} \nmid 2i-1 \text{ and } 0 < i < g/2, \\ \{0, \dots, \lfloor r/2 \rfloor\} & \text{if } g \text{ even and } i = g/2. \end{cases}$$

(ii) Let $\overline{\phi}_{r,d} : \overline{\mathcal{V}ec}_{r,d,g} \to \overline{\mathcal{M}}_g$ be the forgetful map. For $0 \leq i \leq g/2$, we have
$$\overline{\phi}_{r,d}^* \mathcal{O}(\delta_i) = \mathcal{O}\left(\sum_{j \in J_i} \widetilde{\delta}_i^j\right).$$

Proof. Part (i). Observe that $\widetilde{\delta} := \overline{\mathcal{V}ec}_{r,d,g}\backslash\mathcal{V}ec_{r,d,g} = \overline{\phi}_{r,d}^{-1}(\overline{\mathcal{M}}_g\backslash\mathcal{M}_g)$. Clearly, we have a set-theoretically equality

$$\overline{\phi}_{r,d}^{-1}(\delta_i) = \bigcup_{j\in J_i} \widetilde{\delta}_i^j.$$

We can easily see that $\delta_i^j = \delta_t^k$ if and only if $j = k$ and $i = t$. Now we are going to prove that they are irreducible. Let $\widetilde{\delta}^*$ be the locus of $\widetilde{\delta}$ of curves with exactly one node. As in [DM69, Corollary 1.9] we can prove that $\widetilde{\delta}$ is a normal crossing divisor and $\widetilde{\delta}^*$ is a dense smooth open substack in $\widetilde{\delta}$. Moreover, setting $\widetilde{\delta}_i^{*j} := \widetilde{\delta}^* \cap \widetilde{\delta}_i^j$, we see that $\widetilde{\delta}_i^j$ is irreducible if and only if $\widetilde{\delta}_i^{*j}$ is irreducible. It can be shown also that they are disjoint, i.e. $\delta_i^{*j} \cap \delta_t^{*k} \neq \emptyset$ if and only if $j = k$ and $i = t$.
Consider the forgetful map $\phi : \widetilde{\delta}_i^{*j} \to \delta_i^*$, where δ_i^* is the open substack of δ_i of curves with exactly one node. In §1.5, we have seen that the morphism of Artin functors $\mathrm{Def}_{(C,\mathcal{E})} \to \mathrm{Def}_C$ is formally smooth for any nodal curve. This implies that the map ϕ is smooth, in particular is open. Since δ_i^* is irreducible (see [DM69, pag. 94]), it is enough to show that the geometric fibers of ϕ are irreducible.
Let C be a nodal curve with two irreducible components C_1 and C_2, of genus i and $g - i$, meeting at a point x, this defines a geometric point $[C] \in \delta_i^*$. Consider the moduli stack $\widetilde{\delta}_C^j$ of vector bundles on C of multidegree

$$(d_1, d_2) := (\deg_{C_1}\mathcal{E}, \deg_{C_2}\mathcal{E}) = \left(\left\lceil d\frac{2i-1}{2g-2} - \frac{r}{2}\right\rceil + j, \left\lfloor d\frac{2(g-i)-1}{2g-2} + \frac{r}{2}\right\rfloor - j\right).$$

It can be shown that there exists an isomorphism of stacks $\widetilde{\delta}_C^j \to \phi^*([C])$. Observe that defining a properly balanced vector bundle on $\widetilde{\delta}_C^j$ is equivalent to giving a vector bundle on C_1 of degree d_1, a vector bundle on C_2 of degree d_2 and an isomorphism of vector spaces between the fibers at the node. Consider the moduli stack $\mathcal{V}ec_{r,d_1,C_1}$ parametrizing vector bundles on C_1 of degree d_1 and rank r. Let $\mathcal{E}$ be the universal vector bundle on $\mathcal{V}ec_{r,d_1,C_1} \times C_1$. We fix an open (and dense) substack $\mathcal{V}$ such that $\mathcal{E}_{\mathcal{V}\times\{x\}}$ is trivial. Analogously, let $\mathcal{W}$ be an open subset of the moduli stack $\mathcal{V}ec_{r,d_2,C_2}$, parametrizing vector bundles on C_2 of degree d_2 and rank r, such that the universal vector bundle on $\mathcal{V}ec_{r,d_2,C_2} \times C_2$ is trivial along $\mathcal{W} \times \{x\}$. Via glueing procedure, we obtain a dominant morphism $\mathcal{V} \times \mathcal{W} \times GL_r \longrightarrow \widetilde{\delta}_C^j$. The source is irreducible (because $\mathcal{V}$ and $\mathcal{W}$ are irreducible by [Hof10, Corollary A.5]), so the same holds for the target $\widetilde{\delta}_C^j$.
Part (ii). By part (i), for $0 \leq i \leq g/2$ we have

$$\overline{\phi}_{r,d}^*\mathcal{O}(\delta_i) = \mathcal{O}\left(\sum_{j\in J_i} a_i^j \widetilde{\delta}_i^j\right)$$

where a_i^j are integers. We have to prove that the coefficients are 1. We can reduce to prove it locally on $\widetilde{\delta}$. The generic element of $\widetilde{\delta}$ is a pair $(C, \mathcal{E})$ such that C is stable with exactly one node and $\mathrm{Aut}(C, \mathcal{E}) = \mathbb{G}_m$. By Lemma 1.5.3, locally at such $(C, \mathcal{E})$, $\overline{\phi}_{r,d}$ looks like

$$\left[\mathrm{Spf}\, k[\![x_1, \ldots, x_{3g-3}, y_1, \ldots, y_{r^2(g-1)+1}]\!]/\mathbb{G}_m\right] \to \left[\mathrm{Spf}\, k[\![x_1, \ldots, x_{3g-3}]\!]/\mathrm{Aut}(C)\right].$$

We can choose local coordinates such that x_1 corresponds to smoothing the unique node of C. For such a choice of the coordinates, we have that the equation of δ_i locally on C is given by $(x_1 = 0)$ and the equation of $\widetilde{\delta}_i^j$ locally on $(C, \mathcal{E})$ is given by $(x_1 = 0)$. Since $\overline{\phi}_{r,d}^*(x_1) = x_1$, the theorem follows. $\square$

With an abuse of notation we set $\widetilde{\delta}_i^j := \nu_{r,d}(\widetilde{\delta}_i^j)$ for $0 \leq i \leq g/2$ and $j \in J_i$, where $\nu_{r,d} : \overline{\mathcal{V}ec}_{r,d,g} \to \overline{\mathcal{V}}_{r,d,g}$ is the rigidification map. From the above proposition, we deduce the following

Corollary 2.6.3. *The following hold:*

(1) *The boundary $\widetilde{\delta} := \overline{\mathcal{V}}_{r,d,g}/\mathcal{V}ec_{r,d,g}$ of $\overline{\mathcal{V}}_{r,d,g}$ is a normal crossing divisor, and its irreducible components are $\widetilde{\delta}_i^j$ for $0 \leq i \leq g/2$ and $j \in J_i$.*
(2) *For $0 \leq i \leq g/2$, $j \in J_i$ we have $\nu_{r,d}^*\mathcal{O}(\widetilde{\delta}_i^j) = \mathcal{O}(\widetilde{\delta}_i^j)$.*

2.7. **Tautological line bundles.** In this subsection, we will produce several line bundles on the stack $\overline{\mathcal{V}ec}_{r,d,g}$ and we will study their relations in the rational Picard group of $\overline{\mathcal{V}ec}_{r,d,g}$. Consider the universal curve $\overline{\pi}:\overline{\mathcal{V}ec}_{r,d,g,1}\to\overline{\mathcal{V}ec}_{r,d,g}$. The stack $\overline{\mathcal{V}ec}_{r,d,g,1}$ has two natural sheaves, the dualizing sheaf $\omega_{\overline{\pi}}$ and the universal vector bundle $\mathcal{E}$. As explained in §2.2, we can produce the following line bundles which will be called *tautological line bundles*:

$$\begin{array}{rcl} K_{1,0,0} & := & \langle\omega_{\overline{\pi}},\omega_{\overline{\pi}}\rangle, \\ K_{0,1,0} & := & \langle\omega_{\overline{\pi}},\det\mathcal{E}\rangle, \\ K_{-1,2,0} & := & \langle\det\mathcal{E},\det\mathcal{E}\rangle, \\ \Lambda(m,n,l) & := & d_{\overline{\pi}}(\omega_{\overline{\pi}}^m\otimes(\det\mathcal{E})^n\otimes\mathcal{E}^l). \end{array}$$

With an abuse of notation, we will denote with the same symbols their restriction to any open substack of $\overline{\mathcal{V}ec}_{r,d,g}$. By Theorems 2.2.1 and 2.2.2, we can compute the first Chern classes of the tautological line bundles:

$$\begin{array}{rcccl} k_{1,0,0} & := & c_1(K_{1,0,0}) & = & \overline{\pi}_*\left(c_1(\omega_{\overline{\pi}})^2\right), \\ k_{0,1,0} & := & c_1(K_{0,1,0}) & = & \overline{\pi}_*\left(c_1(\omega_{\overline{\pi}})\cdot c_1(\mathcal{E})\right) \\ k_{-1,2,0} & := & c_1(K_{-1,2,0}) & = & \overline{\pi}_*\left(c_1(\mathcal{E})^2\right) \\ \lambda(m,n,l) & := & c_1(\Lambda(m,n,l)) & = & c_1\left(\overline{\pi}_!\left(\omega_{\overline{\pi}}^m\otimes(\det\mathcal{E})^n\otimes\mathcal{E}^l\right)\right) \end{array}$$

Theorem 2.7.1. *The tautological line bundles on $\overline{\mathcal{V}ec}_{r,d,g}$ satisfy the following relations in the rational Picard group $Pic(\overline{\mathcal{V}ec}_{r,d,g})\otimes\mathbb{Q}$.*

(i) $K_{1,0,0}=\Lambda(1,0,0)^{12}\otimes\mathcal{O}(-\widetilde{\delta})$.
(ii) $K_{0,1,0}=\Lambda(1,0,1)\otimes\Lambda(0,0,1)^{-1}=\Lambda(1,1,0)\otimes\Lambda(0,1,0)^{-1}$.
(iii) $K_{-1,2,0}=\Lambda(0,1,0)\otimes\Lambda(1,1,0)\otimes\Lambda(1,0,0)^{-2}$.
(iv) For (m,n,l) integers we have:

$$\begin{aligned} \Lambda(m,n,l) \quad = \quad & \Lambda(1,0,0)^{r^l(6m^2-6m+1-n^2-l)-2r^{l-1}nl-r^{l-2}l(l-1)}\otimes \\ & \otimes\Lambda(0,1,0)^{r^l\left(-mn+\binom{n+1}{2}\right)+r^{l-1}l(n-m)+r^{l-2}\binom{l}{2}}\otimes \\ & \otimes\Lambda(1,1,0)^{r^l\left(mn+\binom{n}{2}\right)+r^{l-1}l(m+n)+r^{l-2}\binom{l}{2}}\otimes \\ & \otimes\Lambda(0,0,1)^{r^{l-1}l}\otimes\mathcal{O}\left(-r^l\binom{m}{2}\widetilde{\delta}\right). \end{aligned}$$

Proof. As we will see in the Lemma 3.1.5, we can reduce to proving the equalities on the quasi-compact open substack $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$. We follow the same strategy in the proof of [MV14, Theorem 5.2]. The first Chern class map is an isomorphism by Theorem 2.1.4. Thus it is enough to prove the above relations in the rational Chow group $A^1\left(\overline{\mathcal{V}ec}^{Pss}_{r,d,g}\right)\otimes\mathbb{Q}$. Applying the Grothendieck-Riemann-Roch Theorem to the universal curve $\overline{\pi}:\overline{\mathcal{V}ec}_{r,d,g,1}\to\overline{\mathcal{V}ec}_{r,d,g}$, we get:

$$\mathrm{ch}\left(\overline{\pi}_!\left(\omega_{\overline{\pi}}^m\otimes(\det\mathcal{E})^n\otimes\mathcal{E}^l\right)\right)=\overline{\pi}_*\left(\mathrm{ch}\left(\omega_{\overline{\pi}}^m\otimes(\det\mathcal{E})^n\otimes\mathcal{E}^l\right)\cdot\mathrm{Td}\left(\Omega_{\overline{\pi}}\right)^{-1}\right) \tag{2.7.1}$$

where ch is the Chern character, Td the Todd class and $\Omega_{\overline{\pi}}$ is the sheaf of relative Kahler differentials. Using Theorem 2.2.1, the degree one part of the left hand side becomes

$$\mathrm{ch}\left(\overline{\pi}_!\left(\omega_{\overline{\pi}}^m\otimes(\det\mathcal{E})^n\otimes\mathcal{E}^l\right)\right)_1=c_1\left(\overline{\pi}_!\left(\omega_{\overline{\pi}}^m\otimes(\det\mathcal{E})^n\otimes\mathcal{E}^l\right)\right)=c_1\left(\Lambda(m,n,l)\right)=\lambda(m,n,l). \tag{2.7.2}$$

In order to compute the right hand side, we will use the fact that $c_1\left(\Omega_{\overline{\pi}}\right)=c_1\left(\omega_{\overline{\pi}}\right)$ and $\overline{\pi}_*\left(c_2\left(\Omega_{\overline{\pi}}\right)\right)=\widetilde{\delta}$ (see [ACG11, p. 383]. Using this, the first three terms of the inverse of the Todd class of $\Omega_{\overline{\pi}}$ are equal to

$$\mathrm{Td}\left(\Omega_{\overline{\pi}}\right)^{-1}=1-\frac{c_1\left(\Omega_{\overline{\pi}}\right)}{2}+\frac{c_1\left(\Omega_{\overline{\pi}}\right)^2+c_2\left(\Omega_{\overline{\pi}}\right)}{12}+\ldots=1-\frac{c_1\left(\omega_{\overline{\pi}}\right)}{2}+\frac{c_1\left(\omega_{\overline{\pi}}\right)^2+c_2\left(\Omega_{\overline{\pi}}\right)}{12}+\ldots \tag{2.7.3}$$

By the multiplicativity of the Chern character, we get

$$
\begin{aligned}
(2.7.4)\quad ch\left(\omega_{\overline{\pi}}^{m}\otimes(\det\mathcal{E})^{n}\otimes\mathcal{E}^{l}\right) &= \mathrm{ch}\left(\omega_{\overline{\pi}}\right)^{m}\mathrm{ch}(\det\mathcal{E})^{n}\mathrm{ch}\left(\mathcal{E}\right)^{l} = \\
&= \left(1+c_1\left(\omega_{\overline{\pi}}\right)+\frac{c_1\left(\omega_{\overline{\pi}}\right)^2}{2}+\ldots\right)^{m}\cdot\left(1+c_1\left(\mathcal{E}\right)+\frac{c_1\left(\mathcal{E}\right)^2}{2}+\ldots\right)^{n}\cdot \\
&\qquad\cdot\left(r+c_1\left(\mathcal{E}\right)+\frac{c_1\left(\mathcal{E}\right)^2-2c_2\left(\mathcal{E}\right)}{2}+\ldots\right)^{l} = \\
&= \left(1+mc_1\left(\omega_{\overline{\pi}}\right)+\frac{m^2}{2}c_1\left(\omega_{\overline{\pi}}\right)^2+\ldots\right)\cdot\left(1+nc_1\left(\mathcal{E}\right)+\frac{n^2}{2}c_1\left(\mathcal{E}\right)^2+\ldots\right)\cdot \\
&\qquad\cdot\left(r^l+lr^{l-1}c_1\left(\mathcal{E}\right)+\frac{lr^{l-2}}{2}\left((r+l-1)c_1\left(\mathcal{E}\right)^2-2rc_2\left(\mathcal{E}\right)\right)\right)+\ldots\right) = \\
&= r^l+\left[rmc_1\left(\omega_{\overline{\pi}}\right)+(rn+l)c_1\left(\mathcal{E}\right)\right]r^{l-1}+\left[r^l\frac{m^2}{2}c_1\left(\omega_{\overline{\pi}}\right)^2+r^{l-1}m\left(rn+l\right)c_1\left(\omega_{\overline{\pi}}\right)c_1\left(\mathcal{E}\right)\right. \\
&\qquad\left.+\frac{r^{l-2}}{2}\left(r^2n^2+lr(2n+1)+l(l-1)\right)c_1\left(\mathcal{E}\right)^2-lr^{l-1}c_2\left(\mathcal{E}\right)\right].
\end{aligned}
$$

Combining (2.7.3) and (2.7.4), we can compute the degree one part of the right hand side of (2.7.1):

$$
\begin{aligned}
(2.7.5)\quad \left[\overline{\pi}_*\left(ch\left(\omega_{\overline{\pi}}^{m}\otimes(\det\mathcal{E})^{n}\otimes\mathcal{E}^{l}\right)\cdot\mathrm{Td}\left(\Omega_{\overline{\pi}}\right)^{-1}\right)\right]_1 &= \overline{\pi}_*\left(\left[ch\left(\omega_{\overline{\pi}}^{m}\otimes(\det\mathcal{E})^{n}\otimes\mathcal{E}^{l}\right)\cdot\mathrm{Td}\left(\Omega_{\overline{\pi}}\right)^{-1}\right]_2\right) = \\
&= \overline{\pi}_*\left(\frac{r^l}{12}(6m^2-6m+1)c_1(\omega_{\overline{\pi}})^2+\frac{r^{l-1}}{2}(rn+l)(2m-1)c_1\left(\omega_{\overline{\pi}}\right)c_1\left(\mathcal{E}\right)+\right. \\
&\qquad\left.+\frac{r^{l-2}}{2}\left(r^2n^2+lr(2n+1)+l(l-1)\right)c_1\left(\mathcal{E}\right)^2-lr^{l-1}c_2\left(\mathcal{E}\right)+\frac{r^l}{12}c_2\left(\Omega_{\overline{\pi}}\right)\right) = \\
&= \frac{r^l}{12}(6m^2-6m+1)k_{1,0,0}+\frac{r^{l-1}}{2}(rn+l)(2m-1)k_{0,1,0}+ \\
&\qquad+\frac{r^{l-2}}{2}\left(r^2n^2+lr(2n+1)+l(l-1)\right)k_{-1,2,0}-lr^{l-1}\overline{\pi}_*c_2\left(\mathcal{E}\right)+\frac{r^l}{12}\widetilde{\delta}.
\end{aligned}
$$

Combining with (2.7.2), we have:

$$
\begin{aligned}
(2.7.6)\quad \lambda(m,n,l) &= \frac{r^l}{12}(6m^2-6m+1)k_{1,0,0}+\frac{r^{l-1}}{2}(rn+l)(2m-1)k_{0,1,0}+ \\
&\qquad+\frac{r^{l-2}}{2}\left(r^2n^2+lr(2n+1)+l(l-1)\right)k_{-1,2,0}-lr^{l-1}\overline{\pi}_*c_2\left(\mathcal{E}\right)+\frac{r^l}{12}\widetilde{\delta}.
\end{aligned}
$$

As special case of the above relation, we get

$$
(2.7.7)\qquad \lambda(1,0,0)=\frac{k_{1,0,0}}{12}+\frac{\widetilde{\delta}}{12}.
$$

If we replace (2.7.7) in (2.7.6), then we have

$$
\begin{aligned}
(2.7.8)\quad \lambda(m,n,l) &= r^l(6m^2-6m+1)\lambda(1,0,0)+\frac{r^{l-1}}{2}(rn+l)(2m-1)k_{0,1,0}+ \\
&\qquad+\frac{r^{l-2}}{2}\left(r^2n^2+lr(2n+1)+l(l-1)\right)k_{-1,2,0}-lr^{l-1}\overline{\pi}_*c_2\left(\mathcal{E}\right)-r^l\binom{m}{2}\widetilde{\delta}.
\end{aligned}
$$

Moreover from (2.7.8) we obtain:

$$
(2.7.9)\qquad \begin{cases}
\lambda(0,1,0)=\lambda(1,0,0)-\frac{k_{0,1,0}}{2}+\frac{k_{-1,2,0}}{2}\\
\lambda(1,1,0)=\lambda(1,0,0)+\frac{k_{0,1,0}}{2}+\frac{k_{-1,2,0}}{2}\\
\lambda(0,0,1)=r\lambda(1,0,0)-\frac{k_{0,1,0}}{2}+\frac{k_{-1,2,0}}{2}-\overline{\pi}_*c_2\left(\mathcal{E}\right)\\
\lambda(1,0,1)=r\lambda(1,0,0)+\frac{k_{0,1,0}}{2}+\frac{k_{-1,2,0}}{2}-\overline{\pi}_*c_2\left(\mathcal{E}\right)
\end{cases}
$$

which gives

$$\begin{cases} k_{0,1,0} = \lambda(1,0,1) - \lambda(0,0,1) = \lambda(1,1,0) - \lambda(0,1,0) \\ k_{-1,2,0} = -2\lambda(1,0,0) + \lambda(0,1,0) + \lambda(1,1,0) \\ \overline{\pi}_* c_2(\mathcal{E}) = (r-1)\lambda(1,0,0) + \lambda(0,1,0) - \lambda(0,0,1). \end{cases} \tag{2.7.10}$$

Substituing in (2.7.8), we finally obtain

$$\begin{aligned} r^{2-l}\lambda(m,n,l) \quad = \quad & \Big(r^2(6m^2-6m+1-n^2-l) - 2rnl - l(l-1)\Big)\lambda(1,0,0) + \\ & + \left(r^2\left(-mn + \binom{n+1}{2}\right) + rl\,(n-m) + \binom{l}{2}\right)\lambda(0,1,0) + \\ & + \left(r^2\left(mn + \binom{n}{2}\right) + rl\,(m+n) + \binom{l}{2}\right)\lambda(1,1,0) + \\ & + rl\lambda(0,0,1) - r^2\binom{m}{2}\widetilde{\delta}. \end{aligned} \tag{2.7.11}$$

□

Remark 2.7.2. As we will see in the next section the integral Picard group of $\mathrm{Pic}(\overline{\mathcal{V}ec}_{r,d,g})$ is torsion free for $g \geq 3$. In particular the relations of Theorem 2.7.1 hold also for $\mathrm{Pic}(\overline{\mathcal{V}ec}_{r,d,g})$.

3. The Picard groups of $\overline{\mathcal{V}ec}_{r,d,g}$ and $\overline{\mathcal{V}}_{r,d,g}$.

The aim of this section is to prove the Theorems A and B. We will prove them in several steps. For the rest of the paper we will assume $r \geq 2$.

3.1. **Independence of the boundary divisors.**

The aim of this subsection is to prove the following

Theorem 3.1.1. *Assume that $g \geq 3$. We have an exact sequence of groups*

$$0 \longrightarrow \bigoplus_{i=0,\ldots,\lfloor g/2 \rfloor} \oplus_{j\in J_i} \langle \mathcal{O}(\widetilde{\delta}_i^j) \rangle \longrightarrow Pic(\overline{\mathcal{V}ec}_{r,d,g}) \longrightarrow Pic(\mathcal{V}ec_{r,d,g}) \longrightarrow 0$$

where the right map is the natural restriction and the left map is the natural inclusion.

For the rest of this subsection, with the only exceptions of Proposition 3.1.2 and Lemma 3.1.11, we will always assume that $g \geq 3$. We recall now a result from [TiB95].

Proposition 3.1.2. [TiB95, Proposition 1.2]. *Let C a nodal curve of genus greater than one without rational components and let $\mathcal{E}$ be a balanced vector bundle over C with rank r and degree d. Let $C_1, \ldots, C_s$ be its irreducible components. If $\mathcal{E}_{C_i}$ is semistable for any i then $\mathcal{E}$ is P-semistable. Moreover if the basic inequalities are all strict and all the $\mathcal{E}_{C_i}$ are semistable and at least one is stable then $\mathcal{E}$ is P-stable.*

Remark 3.1.3. Recall that for a smooth curve of genus greater than 1 the generic vector bundle is stable. On the other hand for an elliptic curve the stable locus is not empty if and only if the degree and the rank are coprime. In this case any semistable vector bundle is stable. In general for an elliptic curve the generic vector bundle of degree d and rank r is direct sum of $n_{r,d}$ stable vector bundles of degree $d/n_{r,d}$ and rank $r/n_{r,d}$; in particular it will be semistable.

We deduce from this

Lemma 3.1.4. *The generic point of $\widetilde{\delta}_i^j$ is a curve C with exactly one node and a properly balanced vector bundle $\mathcal{E}$ such that*

(i) if $i = 0$ the pull-back of $\mathcal{E}$ at the normalization is a stable vector bundle,
(ii) if $i = 1$ the restriction $\mathcal{E}_{C_1}$ is direct sum of stable vector bundles with same rank and degree and $\mathcal{E}_{C_2}$ is a stable vector bundle.
(iii) if $2 \leq i \leq \lfloor g/2 \rfloor$ the restrictions $\mathcal{E}_{C_1}$ and $\mathcal{E}_{C_2}$ are stable vector bundles.

Furthermore the generic point of $\widetilde{\delta}_i^j$ is a curve with exactly one node with a P-stable vector bundle if $\widetilde{\delta}_i^j$ is a non-extremal divisor and a curve with exactly one node with a strictly P-semistable vector bundle if $\widetilde{\delta}_i^j$ is an extremal divisor.

Proof. The case $i=0$ is obvious. We fix $i\in\{1,\dots,\lfloor g/2\rfloor\}$ and $j\in J_i$. By definition the generic point of $\widetilde{\delta}_i^j$ is a curve with two irreducible components C_1 and C_2 of genus i and $g-i$ meeting at one point and a vector bundle $\mathcal{E}$ with multidegree

$$(\deg_{C_1}\mathcal{E},\deg_{C_2}\mathcal{E})=\left(\left\lceil d\frac{2i-1}{2g-2}-\frac{r}{2}\right\rceil+j,\left\lfloor d\frac{2(g-i)-1}{2g-2}+\frac{r}{2}\right\rfloor-j\right).$$

As observed in Remark 3.1.3 the generic vector bundle over a smooth curve of genus >1 (resp. 1) is stable (resp. direct sum of stable vector bundles). Giving a vector bundle over C is equivalent to give a vector bundle on any irreducible component and an isomorphism of vector spaces between the fibers at the nodes. With this in mind, it is easy to see that we can deform any vector bundle $\mathcal{E}$ in a vector bundle $\mathcal{E}'$ which is stable (resp. is a direct sum of stable vector bundles with same rank and degree) over any component of genus >1 (resp. 1). By Proposition 3.1.2, the generic point of $\widetilde{\delta}_i^j$ is P-semistable. Moreover if $\widetilde{\delta}_i^j$ is a non-extremal divisor the basic inequalities are strict. By the second assertion of *loc. cit.*, if $\widetilde{\delta}_i^j$ is a non-extremal divisor the generic point of $\widetilde{\delta}_i^j$ is P-stable. It remains to prove the assertion for the extremal divisors. Suppose that $\widetilde{\delta}_i^0$ is an extremal divisor, the proof for the $\widetilde{\delta}_i^r$ is similar. It is easy to prove that

$$\deg_{C_1}\mathcal{E}=d\frac{2i-1}{2g-2}-\frac{r}{2}\iff\frac{\chi(\mathcal{E}_{C_1})}{\omega_{C_1}}=\frac{\chi(\mathcal{E})}{\omega_C}.$$

In other words, $\mathcal{E}_{C_1}$ is a destabilizing quotient for $\mathcal{E}$, concluding the proof. □

Lemma 3.1.5. *The Picard group of $\mathcal{V}ec_{r,d,g}$ (resp. $\mathcal{V}_{r,d,g}$), is naturally isomorphic to the Picard group of the open substacks $\mathcal{V}ec^{ss}_{r,d,g}$ (resp. $\mathcal{V}^{ss}_{r,d,g}$) and $\mathcal{U}_n$ (resp. $\mathcal{U}_n\,/\!\!/\mathbb{G}_m$) for n big enough.*
The Picard group of $\overline{\mathcal{V}ec}_{r,d,g}$ (resp. $\overline{\mathcal{V}}_{r,d,g}$), is naturally isomorphic to the Picard group of the open substacks $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ (resp. $\overline{\mathcal{V}}^{Pss}_{r,d,g}$) and $\overline{\mathcal{U}}_n$ (resp. $\overline{\mathcal{U}}_n\,/\!\!/\mathbb{G}_m$) for n big enough.

Proof. We have the following equalities

$$\begin{aligned}\dim\mathcal{V}ec_{r,d,g}&=\dim\mathcal{J}ac_{d,g}+\dim\mathcal{V}ec_{=\mathcal{L},C},\\ \dim\left(\mathcal{V}ec_{r,d,g}\backslash\mathcal{V}ec^{ss}_{r,d,g}\right)&\le\dim\mathcal{J}ac_{d,g}+\dim\left(\mathcal{V}ec_{=\mathcal{L},C}\backslash\mathcal{V}ec^{ss}_{=\mathcal{L},C}\right).\end{aligned}$$

Thus $\mathrm{cod}(\mathcal{V}ec_{r,d,g}\backslash\mathcal{V}ec^{ss}_{r,d,g},\mathcal{V}ec_{r,d,g})\ge\mathrm{cod}(\mathcal{V}ec_{=\mathcal{L},C}\backslash\mathcal{V}ec^{ss}_{=\mathcal{L},C},\mathcal{V}ec_{=\mathcal{L},C})\ge 2$ (see proof of [Hof12, Corollary 3.2]). By Proposition 1.3.2, there exists $n_*\gg 0$ such that $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}\subset\overline{\mathcal{U}}_n$ for $n\ge n_*$. In particular $\mathrm{cod}(\mathcal{U}_n\backslash\mathcal{V}ec^{ss}_{r,d,g},\mathcal{U}_n)\ge 2$. Suppose that $\mathrm{cod}(\overline{\mathcal{V}ec}_{r,d,g}\backslash\overline{\mathcal{V}ec}^{Pss}_{r,d,g},\overline{\mathcal{V}ec}_{r,d,g})=1$, so $\overline{\mathcal{V}ec}_{r,d,g}\backslash\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$ contains a substack of codimension 1. By the observations above this stack must be contained in some irreducible components of $\widetilde{\delta}$. The generic point of any divisor $\widetilde{\delta}_i^j$ is P-semistable by Lemma 3.1.4, then we have a contradiction. So $\mathrm{cod}(\overline{\mathcal{V}ec}_{r,d,g}\backslash\overline{\mathcal{V}ec}^{Pss}_{r,d,g},\overline{\mathcal{V}ec}_{r,d,g})\ge\mathrm{cod}(\overline{\mathcal{U}}_n\backslash\overline{\mathcal{V}ec}^{Pss}_{r,d,g},\overline{\mathcal{U}}_n)\ge 2$. The same holds for the rigidifications. By Theorem 2.1.4, the lemma follows. □

By Lemma 3.1.5, Theorem 3.1.1 is equivalent to proving that there exists $n_*\gg 0$ such that for $n\ge n_*$ we have an exact sequence of groups

$$0\longrightarrow\bigoplus_{i=0,\dots,\lfloor g/2\rfloor}\oplus_{j\in J_i}\langle\mathcal{O}(\widetilde{\delta}_i^j)\rangle\longrightarrow\mathrm{Pic}(\overline{\mathcal{U}}_n)\longrightarrow\mathrm{Pic}(\mathcal{U}_n)\longrightarrow 0.$$

By Theorem 2.1.4, the sequence exists and it is exact in the middle and at right. It remains to prove the left exactness. The strategy that we will use is the same as the one of Arbarello-Cornalba for $\overline{\mathcal{M}}_g$ in [AC87] and the generalization for $\overline{\mathcal{J}ac}_{r,g}$ done by Melo-Viviani in [MV14]. More precisely, we will construct morphisms $B\to\overline{\mathcal{U}}_n$ from irreducible smooth projective curves B and we compute the degree of the pull-backs of the boundary divisors of $\mathrm{Pic}(\overline{\mathcal{U}}_n)$ to B. We will construct liftings of the families F_h (for $1\le h\le(g-2)/2$), F and F' used by Arbarello-Cornalba in [AC87, pp. 156-159]. Since $\overline{\mathcal{V}ec}_{r,d,g}\cong\overline{\mathcal{V}ec}_{r,d',g}$ if $d\equiv d'$ mod $(r(2g-2))$, in this section we can assume that $0\le d<r(2g-2)$.

The Family $\widetilde{F}$.

Consider a general pencil in the linear system $H^0(\mathbb{P}^2,\mathcal{O}(2))$. It defines a rational map $\mathbb{P}^2\dashrightarrow\mathbb{P}^1$, which is regular outside of the four base points of the pencil. Blowing the base locus we get a conic bundle $\phi:X\to\mathbb{P}^1$. The four exceptional divisors E_1, E_2, E_3, $E_4\subset X$ are sections of ϕ. It can be shown that the conic bundle

has 3 singular fibers consisting of rational chains of length two. Fix a smooth curve C of genus $g-3$ and p_1, p_2, p_3, p_4 points of C. Consider the following surface

$$Y = \left(X \amalg (C \times \mathbb{P}^1)\right)/(E_i \sim \{p_i\} \times \mathbb{P}^1).$$

We get a family $f : Y \to \mathbb{P}^1$ of stable curves of genus g. The general fiber of f is as in Figure 1 where Q is a smooth conic.

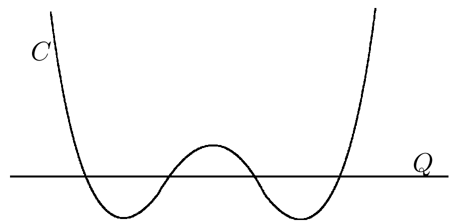

FIGURE 1. The general fiber of $f : Y \to \mathbb{P}^1$

While the 3 special components are as in Figure 2 where R_1 and R_2 are rational curves.

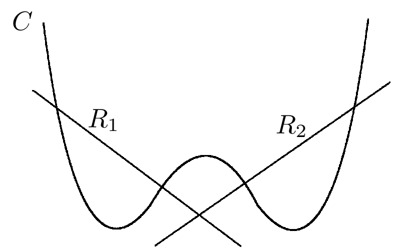

FIGURE 2. The three special fibers of $f : Y \to \mathbb{P}^1$

Choose a vector bundle of degree d on C, pull it back to $C \times \mathbb{P}^1$ and call it E. Since E is trivial on $\{p_i\} \times \mathbb{P}^1$, we can glue it with the trivial vector bundle of rank r on X obtaining a vector bundle $\mathcal{E}$ on $f : Y \to \mathbb{P}^1$ of relative rank r and degree d.

Lemma 3.1.6. *$\mathcal{E}$ is properly balanced.*

Proof. $\mathcal{E}$ is obviously admissible because is defined over a family of stable curves. Since being properly balanced is an open condition, we can reduce to check that $\mathcal{E}$ is properly balanced on the three special fibers. By Remark 1.1.15, it is enough to check the basic inequality for the subcurves $R_1 \cup R_2$, R_1 and R_2. And by the assumption $0 \le d < r(2g-2)$ is easy to see that the inequalities holds. □

We call $\widetilde{F}$ the family $f : X \to \mathbb{P}^1$ with the vector bundle $\mathcal{E}$. It is a lifting of the family F defined in [AC87, p. 158]. So we can compute the degree of the pull-backs of the boundary bundles in $\mathrm{Pic}(\overline{\mathcal{V}ec}_{r,d,g})$ to the curve $\widetilde{F}$. Consider the commutative diagram

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\widetilde{F}"] \arrow[dr, "F"'] & \overline{\mathcal{V}ec}_{r,d,g} \arrow[d, "\overline{\phi}_{r,d}"] \\ & \overline{\mathcal{M}}_g \end{tikzcd}$$

By Proposition 2.6.2, we have $\deg_{\widetilde{F}} \mathcal{O}(\widetilde{\delta}_0) = \deg_F \mathcal{O}(\delta_0)$ and $\deg_F \mathcal{O}(\delta_0) = -1$ by [AC87, p. 158]. Since $\widetilde{F}$ does not intersect the other boundary divisors, we have:

$$\begin{cases} \deg_{\widetilde{F}} \mathcal{O}(\widetilde{\delta}_0) = -1, & \\ \deg_{\widetilde{F}} \mathcal{O}(\widetilde{\delta}_i^j) = 0 & \text{if } i \neq 0 \text{ and } j \in J_i. \end{cases}$$

The Families $\widetilde{F}_1'^j$ and $\widetilde{F}_2'^j$ (for $j \in J_1$).
We start with the same family of conics $\phi : X \to \mathbb{P}^1$ and the same smooth curve C used for the family $\widetilde{F}$. Let Γ be a smooth elliptic curve and take points $p_1 \in \Gamma$ and $p_2, p_3, p_4 \in C$. We construct a new surface

$$Z = \left(X \amalg (C \times \mathbb{P}^1) \amalg (\Gamma \times \mathbb{P}^1)\right) / (E_i \sim \{p_i\} \times \mathbb{P}^1).$$

We obtain a family $g : Z \to \mathbb{P}^1$ of stable curves of genus g. The general fiber is as in Figure 3 where Q is a smooth conic. The three special fibers are as in Figure 4 where R_1 and R_2 are rational smooth curves.

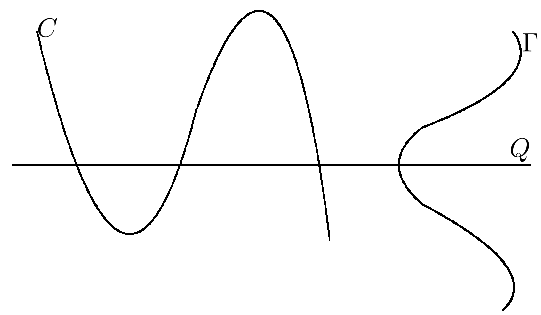

FIGURE 3. The general fibers of $g : Z \to \mathbb{P}^1$.

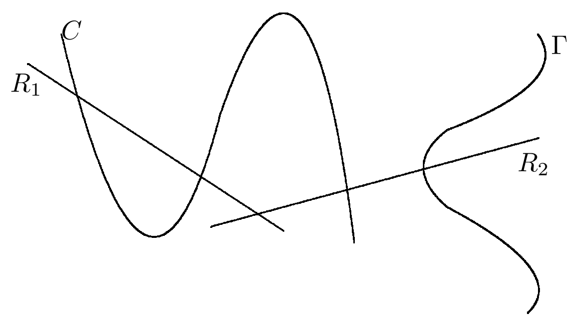

FIGURE 4. The three special fibers of $g : Z \to \mathbb{P}^1$.

Let j be an integer. We choose two vector bundles of degree $d-j$ and $d-3j$ on C, pull them back to $C \times \mathbb{P}^1$ and call them G_1^j and G_2^j. We choose a vector bundle of degree j on Γ, pull it back to $\Gamma \times \mathbb{P}^1$ and call it M^j. We glue the vector bundle G_1^j (resp. G_2^j) on $C \times \mathbb{P}^1$, the vector bundle M^j on $\Gamma \times \mathbb{P}^1$ and the vector bundle $\mathcal{O}_X^r$ (resp. $\phi^*\mathcal{O}_{\mathbb{P}^1}(j) \otimes \omega_{X/\mathbb{P}^1}^{-j} \oplus \mathcal{O}_X^{r-1}$), obtaining a vector bundle $\mathcal{G}_1^j$ (resp. $\mathcal{G}_2^j$) on Z of relative rank r and degree d.

Lemma 3.1.7. *Let j be an integer such that*

$$\left| j - \frac{d}{2g-2} \right| \leq \frac{r}{2}$$

Then $\mathcal{G}_1^j$ is properly balanced if $0 \leq d \leq r(g-1)$ and $\mathcal{G}_2^j$ is properly balanced if $r(g-1) \leq d < r(2g-2)$.

Proof. As before we can check the condition on the special fibers. By Remark 1.1.15 we can reduce to check the inequalities for the subcurves Γ, C, R_1 and $R_2 \cup \Gamma$. Suppose that $0 \leq d \leq r(g-1)$ and consider $\mathcal{G}_1^j$. The inequality on Γ follows by hypothesis. The inequality on C is

$$\left| d - j - d\frac{2g-5}{2g-2} \right| \leq \frac{3}{2}r \iff \left| j - d\frac{3}{2g-2} \right| \leq \frac{3}{2}r,$$

and this follows by these inequalities (true by hypothesis on j and d)

$$\left| j - d\frac{3}{2g-2} \right| \leq \left| j - \frac{d}{2g-2} \right| + \left| \frac{d}{g-1} \right| \leq \frac{r}{2} + r.$$

The inequality on R_1 is
$$\left|\frac{d}{2g-2}\right| \le \frac{3}{2}r,$$
and this follows by the hypothesis on d. Finally the inequality on $R_2 \cup \Gamma$ is
$$\left|j - \frac{d}{g-1}\right| \le r,$$
and this follows by the following inequalities (true by hypothesis on j and d)
$$\left|j - \frac{d}{g-1}\right| \le \left|j - \frac{d}{2g-2}\right| + \left|\frac{d}{2g-2}\right| \le \frac{r}{2} + \frac{r}{2}.$$
Suppose next that $r(g-1) \le d < r(2g-2)$ and consider $\mathcal{G}_2^j$. The inequality on Γ follows by hypothesis. On C, the inequality gives
$$\left|d - 3j - d\frac{2g-5}{2g-2}\right| \le \frac{3}{2}r \iff \left|j - \frac{d}{2g-2}\right| \le \frac{r}{2},$$
which follows by hypothesis on j. The inequality on R_1 is
$$\left|j - \frac{d}{2g-2}\right| \le \frac{3}{2}r,$$
and this follows by hypothesis on j. The inequality on $R_2 \cup \Gamma$ is
$$\left|2j - \frac{d}{g-1}\right| \le r,$$
and this follows by the inequalities (true by hypothesis on j)
$$\left|2j - \frac{d}{g-1}\right| \le 2\left|j - \frac{d}{2g-2}\right| \le r.$$
□

Let $k \in J_1$. If $0 \le d \le r(g-1)$, we call $\widetilde{F'}_1^k$ the family $g : Z \to \mathbb{P}^1$ with the properly balanced vector bundle $\mathcal{G}_1^{\lceil \frac{d}{2g-2} - \frac{r}{2}\rceil + k}$. If $r(g-1) \le d < r(2g-2)$ we call $\widetilde{F'}_2^k$ the family $g : Z \to \mathbb{P}^1$ with the properly balanced vector bundle $\mathcal{G}_2^{\lceil \frac{d}{2g-2} - \frac{r}{2}\rceil + k}$. As before we compute the degree of boundary line bundles to the curves $\widetilde{F'}_1^k$ and $\widetilde{F'}_2^k$ (in the range of degrees where they are defined) using the fact that they are liftings of the family F' in [AC87, p. 158]. If $0 \le d \le r(g-1)$ then we have
$$\begin{cases} \deg_{\widetilde{F'}_1^k} \mathcal{O}(\widetilde{\delta}_1^k) = -1, & \\ \deg_{\widetilde{F'}_1^k} \mathcal{O}(\widetilde{\delta}_1^j) = 0 & \text{if } j \ne k, \\ \deg_{\widetilde{F'}_1^k} \mathcal{O}(\widetilde{\delta}_i^j) = 0 & \text{if } i > 1, \text{ for any } j \in J_i. \end{cases}$$
Indeed the first two relations follow from
$$\deg_{\widetilde{F'}_1^k} \mathcal{O}\left(\sum_{j \in J_i} \widetilde{\delta}_1^j\right) = \deg_{F'} \mathcal{O}(\delta_1) = -1$$
(see [AC87, p. 158]) and the fact that $\widetilde{F'}_1^k$ does not meet $\widetilde{\delta}_1^j$ for $k \ne j$. The last follows by the fact that $\widetilde{F'}_1^k$ does not meet $\widetilde{\delta}_i^j$ for $i > 1$. Similarly for $\widetilde{F'}_2^k$ we can show that for $r(g-1) \le d < r(2g-2)$, we have
$$\begin{cases} \deg_{\widetilde{F'}_2^k} \mathcal{O}(\delta_1^k) = -1, & \\ \deg_{\widetilde{F'}_2^k} \mathcal{O}(\delta_1^j) = 0 & \text{if } j \ne k, \\ \deg_{\widetilde{F'}_2^k} \mathcal{O}(\delta_i^j) = 0 & \text{if } i > 1. \end{cases}$$

The Families $\widetilde{F}_h^j$ (for $1 \le h \le \frac{g-2}{2}$ and $j \in J_h$).
Consider smooth curves C_1, C_2 and Γ of genus h, $g-h-1$ and 1, respectively, and points $x_1 \in C_1$, $x_2 \in C_2$ and $\gamma \in \Gamma$. Consider the surface Y_2 given by the blow-up of $\Gamma \times \Gamma$ at (γ, γ). Let $p_2 : Y_2 \to \Gamma$ be the map

given by composing the blow-down $Y_2 \to \Gamma \times \Gamma$ with the second projection, and $\pi_1 : C_1 \times \Gamma \to \Gamma$ and $\pi_3 : C_2 \times \Gamma \to \Gamma$ be the projections along the second factor. As in [AC87, p. 156] (and [MV14]), we set (see also Figure 5):

$$\begin{aligned}
A &= \{x_1\} \times \Gamma,\\
B &= \{x_2\} \times \Gamma,\\
E &= \text{ exceptional divisor of the blow-up of } \Gamma \times \Gamma \text{ at } (\gamma, \gamma),\\
\Delta &= \text{ proper transform of the diagonal in } Y_2,\\
S &= \text{ proper transform of } \{\gamma\} \times \Gamma \text{ in } Y_2,\\
T &= \text{ proper transform of } \Gamma \times \{\gamma\} \text{ in } Y_2.
\end{aligned}$$

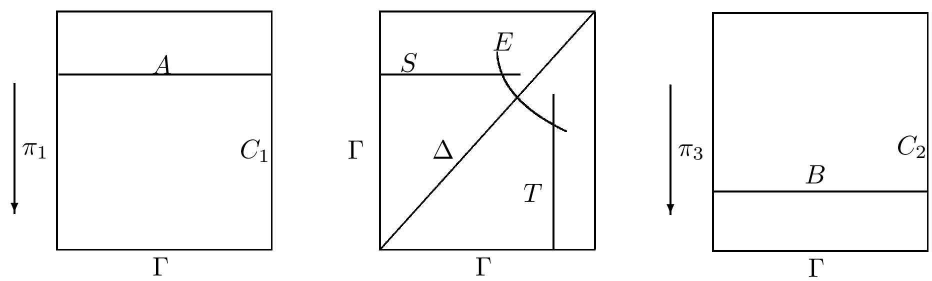

Figure 5. Constructing $f : X \to \Gamma$.

Consider the line bundles $\mathcal{O}_{Y_2}$, $\mathcal{O}_{Y_2}(\Delta)$, $\mathcal{O}_{Y_2}(E)$ over the surface Y_2. From [MV14, p. 16-17], we obtain the Table 1.

	deg_E	deg_T	restriction to Δ	restriction to S
$\mathcal{O}_{Y_2}$	0	0	$\mathcal{O}_\Gamma$	$\mathcal{O}_\Gamma$
$\mathcal{O}_{Y_2}(\Delta)$	1	0	$\mathcal{O}_\Gamma(-\gamma)$	$\mathcal{O}_\Gamma$
$\mathcal{O}_{Y_2}(E)$	-1	1	$\mathcal{O}_\Gamma(\gamma)$	$\mathcal{O}_\Gamma(\gamma)$

Table 1

We construct a surface X by identifying S with A and Δ with B. The surface X comes equipped with a projection $f : X \to \Gamma$. The fibers over all the points $\gamma' \neq \gamma$ are shown in Figure 6, while the fiber over the point γ is shown in Figure 7.

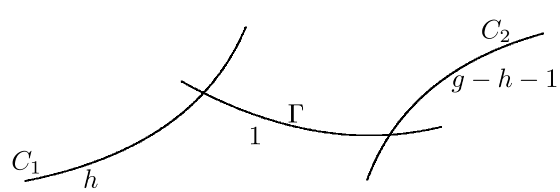

Figure 6. The general fiber of $f : X \to \Gamma$.

Let j, k, t be integers. Consider a vector bundle on C_1 of rank $r-1$ and degree j, we pull-back it on $C_1 \times \Gamma$

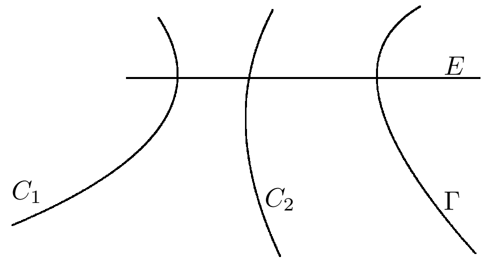

FIGURE 7. The special fiber of $f : X \to \Gamma$.

and call it H^j. Similarly consider a vector bundle on C_2 of rank $r-1$ and degree k, we pull-back it on $C_2 \times \Gamma$ and call it P^k. Consider the following vector bundles

$$\begin{array}{lll} M^{j,k,t}_{C_1\times\Gamma} := H^j \oplus \pi_1^*\mathcal{O}_\Gamma(t\gamma) & \text{on} & C_1 \times \Gamma, \\ M^{j,k,t}_{C_2\times\Gamma} := P^k \oplus \pi_3^*\mathcal{O}_\Gamma(j+k+t-d)\gamma) & \text{on} & C_2 \times \Gamma, \\ M^{j,k,t}_{Y_2} := \mathcal{O}^{r-1}_{Y_2} \oplus \mathcal{O}_{Y_2}((d-j-k)\Delta + tE)) & \text{on} & Y_2. \end{array}$$

By Table 1 we have $M^{j,k,t}_{C_1\times\Gamma}|_A \cong M^{j,k,t}_{Y_2}|_S$ and $M^{j,k,t}_{C_2\times\Gamma}|_B \cong M^{j,k,t}_{Y_2}|_\Delta$. So we can glue the vector bundles in a vector bundle $\mathcal{M}^{j,k,t}_h$ on the family $f : X \to \Gamma$. Moreover, by Table 1, on the special fiber we have

$$\begin{cases} \deg_{C_1}(\mathcal{M}^{j,k,t}_h|_{f^{-1}(\gamma)}) = \deg_{\pi_1^{-1}(\gamma)}(M^{j,k,t}_{C_1\times\Gamma}) = j, \\ \deg_{C_2}(\mathcal{M}^{j,k,t}_h|_{f^{-1}(\gamma)}) = \deg_{\pi_3^{-1}(\gamma)}(M^{j,k,t}_{C_2\times\Gamma}) = k, \\ \deg_{\Gamma}(\mathcal{M}^{j,k,t}_h|_{f^{-1}(\gamma)}) = \deg_T(M^{j,k,t}_{Y_2}) = t, \\ \deg_E(\mathcal{M}^{j,k,t}_h|_{f^{-1}(\gamma)}) = \deg_E(M^{j,k,t}_{Y_2}) = d-j-k-t. \end{cases}$$

In particular $\mathcal{M}^{j,k,t}_h$ has relative degree d.

Lemma 3.1.8. *If j, k, t satisfies:*

$$\left|j - d\frac{2h-1}{2g-2}\right| \leq \frac{r}{2};\ \left|k - d\frac{2g-2h-3}{2g-2}\right| \leq \frac{r}{2};\ \left|t - \frac{d}{2g-2}\right| \leq \frac{r}{2}$$

then $\mathcal{M}^{j,k,t}_h$ is properly balanced.

Proof. We can reduce to check the condition just on the special fiber. By Remark 1.1.15, it is enough to check the inequalities on C_1, C_2 and Γ; this follows easily from the numerical assumptions. □

For any $1 \leq h \leq \frac{g-2}{2}$ choose $j(h)$, resp. $t(h)$, satisfying the first, resp. third, inequality of lemma (observe that such numbers are not unique in general). For every $k \in J_{h+1}$ we call $\widetilde{F}^k_h$ the family $f : X \to \Gamma$ with the properly balanced vector bundle

$$\mathcal{M}_h^{j(h),\lfloor d\frac{2g-2h-3}{2g-2}+\frac{r}{2}\rfloor-k,t(h)}.$$

As before we compute the degree of the boundary line bundles to the curves $\widetilde{F}^k_h$ using the fact that they are liftings of families F_h of [AC87, p. 156]. We get

$$\begin{cases} \deg_{\widetilde{F}^k_h} \mathcal{O}(\widetilde{\delta}^k_{h+1}) = -1, & \\ \deg_{\widetilde{F}^k_h} \mathcal{O}(\widetilde{\delta}^j_{h+1}) = 0 & \text{if } j \neq k, \\ \deg_{\widetilde{F}^k_h} \mathcal{O}(\delta^j_i) = 0 & \text{if } h+1 < i, \text{ for any } j \in J_i. \end{cases}$$

Indeed, the first two relations follow by

$$\deg_{\widetilde{F}^k_h} \mathcal{O}\left(\sum_{j\in J_{h+1}} \widetilde{\delta}^j_{h+1}\right) = \deg_{F_h} \mathcal{O}(\delta_{h+1}) = -1$$

(see [AC87, p. 157]) and the fact that $\widetilde{F}^k_h$ does not meet $\widetilde{\delta}^j_{h+1}$ for $j \neq k$. The last follows from the fact $\widetilde{F}^k_h$ does not meet $\widetilde{\delta}^j_i$ for $i > h+1$.

Proof of Theorem 3.1.1. We know that there exists n_* such that $\overline{\mathcal{V}ec}_{r,d,g}$ and $\overline{\mathcal{U}}_n$ have the same Picard groups for $n \geq n_*$. We can suppose n_* big enough such that families constructed before define curves in $\overline{\mathcal{U}}_n$ for $n \geq n_*$. Suppose that there exists a linear relation

$$\mathcal{O}\left(\sum_i \sum_{j\in J_i} a_i^j \widetilde{\delta}_i^j\right) \cong \mathcal{O} \in \mathrm{Pic}(\overline{\mathcal{U}}_n)$$

where a_i^j are integers. Pulling back to the curve $\widetilde{F} \to \overline{\mathcal{U}}_n$ we deduce $a_0 = 0$. Pulling back to the curves $\widetilde{F'}^j_1 \to \overline{\mathcal{U}}_n$ and $\widetilde{F'}^j_2 \to \overline{\mathcal{U}}_n$ (in the range of degrees where they are defined) we deduce $a_1^j = 0$ for any $j \in J_1$. Pulling back to the curve $\widetilde{F}_h^j \to \overline{\mathcal{U}}_n$ we deduce $a_{h+1}^j = 0$ for any $j \in J_{h+1}$ and $1 \leq h \leq \frac{g-2}{2}$. This concludes the proof. □

We have a similar result for the rigidified stack $\overline{\mathcal{V}}_{r,d,g}$.

Corollary 3.1.9. *We have an exact sequences of groups*

$$0 \longrightarrow \bigoplus_{i=0,\ldots,\lfloor g/2\rfloor} \oplus_{j\in J_i}\langle \mathcal{O}(\widetilde{\delta}_i^j)\rangle \longrightarrow Pic(\overline{\mathcal{V}}_{r,d,g}) \longrightarrow Pic(\mathcal{V}_{r,d,g}) \longrightarrow 0$$

where the right map is the natural restriction and the left map is the natural inclusion.

Proof. As before the only thing to prove is the independence of the boundary line bundles in $\mathrm{Pic}(\overline{\mathcal{V}}_{r,d,g})$. By Theorem 3.1.1 and Corollary 2.6.3, we can reduce to prove the injectivity of $\nu_{r,d}^*: \mathrm{Pic}(\overline{\mathcal{V}}_{r,d,g}) \to \mathrm{Pic}(\overline{\mathcal{V}ec}_{r,d,g})$. A quick way to prove this it is using the Leray spectral sequence associated to the rigidification morphism $\nu_{r,d}: \overline{\mathcal{V}ec}_{r,d,g} \to \overline{\mathcal{V}}_{r,d,g}$ as in the §3.3. □

Remark 3.1.10. As observed before we have that the boundary line bundles are independent on the Picard groups of $\overline{\mathcal{V}ec}_{r,d,g}$, $\overline{\mathcal{V}ec}^{Pss}_{r,d,g}$, $\overline{\mathcal{V}}_{r,d,g}$, $\overline{\mathcal{V}}^{Pss}_{r,d,g}$. A priori we do not know if $\widetilde{\delta}_i^j$ is a divisor of $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$ for any i and $j \in J_i$, because it can be difficult to check when a point $(C,\mathcal{E})$ is H-semistable if C is singular. But as explained in Remark 1.3.3, if $(C,\mathcal{E})$ is P-stable then it is also H-stable. By Proposition 3.1.2, we know that if $\widetilde{\delta}_i^j$ is a non-extremal divisor the generic point of $\widetilde{\delta}_i^j$ is P-stable, in particular it is H-stable.

The end of the section is devoted to prove that also the extremal divisors are in $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$, more precisely the generic points of the extremal divisors in $\overline{\mathcal{V}ec}_{r,d,g}$ are strictly H-semistable. To this aim, we will use the following criterion to prove strictly H-semistability.

Lemma 3.1.11. *Assume that $g \geq 2$. Let $(C,\mathcal{E}) \in \overline{\mathcal{V}ec}_{r,d,g}$ such that C has two irreducible smooth components C_1 and C_2 of genus $1 \leq g_{C_1} \leq g_{C_2}$ meeting at N points $p_1,\ldots,p_N$. Suppose that $\mathcal{E}_{C_1}$ is direct sum of stable vector bundles with the same rank q and same degree e such that e/q is equal to the slope of $\mathcal{E}_{C_1}$ and $\mathcal{E}_{C_2}$ is a stable vector bundle. If $\mathcal{E}$ has multidegree*

$$(deg_{C_1}\mathcal{E}, deg_{C_2}\mathcal{E}) = \left(d\frac{\omega_{C_1}}{\omega_C} - N\frac{r}{2}, d\frac{\omega_{C_2}}{\omega_C} + N\frac{r}{2}\right) \in \mathbb{Z}^2$$

then $(C,\mathcal{E})$ is strictly P-semistable and strictly H-semistable.

Proof. By Proposition 3.1.2, $\mathcal{E}$ is P-semistable. We observe that multidegree condition is equivalent to $\omega_C\chi(\mathcal{E}_{C_1}) = \omega_{C_1}\chi(\mathcal{E})$, so $\mathcal{E}$ is strictly P-semistable. Suppose that $\mathcal{M}$ is a destabilizing subsheaf of $\mathcal{E}$ of multirank (m_1, m_2). Consider the exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E}_{C_2}(-\sum_1^N p_i) \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{E}_{C_1} \arrow[r] & 0 \\
0 \arrow[r] & \mathcal{M}_2 \arrow[r] \arrow[u, hook] & \mathcal{M} \arrow[r] \arrow[u, hook] & \mathcal{M}_1 \arrow[r] \arrow[u, hook] & 0
\end{tikzcd}$$

From this we have

$$\chi(\mathcal{M}) = \chi(\mathcal{M}_1) + \chi(\mathcal{M}_2) \leq \frac{m_1}{r}\chi(\mathcal{E}_{C_1}) + \frac{m_2}{r}\chi(\mathcal{E}_{C_2}(-\sum p_i)) = \frac{m_1\omega_{C_1} + m_2\omega_{C_2}}{r\omega_C}\chi(\mathcal{E}).$$

By hypothesis, $\mathcal{E}_{C_2}$ stable. So we have two possibilities: $\mathcal{M}_2$ is 0 or $\mathcal{E}_{C_2}(-\sum_1^N p_i)$, because otherwise the inequality above is strict. Suppose that $\mathcal{M}_2 = 0$. Then $\mathcal{M} = \mathcal{M}_1$ which implies that $\mathcal{M} \subset \mathcal{E}_{C_1}(-\sum p_i)$ so the inequality above is strict. Thus we have just one possibility: if $\mathcal{M}$ is destabilizing sheaf then $\mathcal{E}_{C_2}(-\sum_1^N p_i) \subset \mathcal{M}$.

In [Sch04, §2.2] there is the following criterion to check if a point is H-semistable. A point $(C,\mathcal{E})$ is H-semistable if and only if $(C^{st}, \pi_*\mathcal{E})$ is P-semistable and for any one-parameter subgroup λ such that $\mathcal{E}$ is strictly P-semistable for λ then $(C,\mathcal{E})$ is Hilbert-semistable for λ. Observe that, in our case, $(C,\mathcal{E}) = (C^{st}, \pi_*\mathcal{E})$.

Let n be a natural number big enough such that $\overline{\mathcal{V}ec}^{Pss}_{r,d,g} \subset \overline{\mathcal{U}}_n$, set $V_n := H^0(C, \mathcal{E}(n))$ and let $B_n := \{v_1, \ldots, v_{\dim V_n}\}$ be a basis for V_n such that λ is given with respect to this basis by the weight vector

$$\sum_{i=1}^{\dim V_n - 1} \alpha_i(\underbrace{i - \dim V_n, \ldots, i - \dim V_n}_{i}, \underbrace{i, \ldots, i}_{i - \dim V_n})$$

where α_i are non-negative rational numbers. $\mathcal{E}$ is strictly P-semistable with respect to λ if and only if there exists a chain of subsheaves $\mathcal{F}_1 \subset \ldots \subset \mathcal{F}_k$ such that

- $\sum_i^k \alpha_i \left(\chi(\mathcal{E}(n)) \left(\sum_j rk\left(\mathcal{F}_{i|_{C_j}}\right)\omega_{C_j}\right) - \chi(\mathcal{F}_i(n)) r\omega_C\right) = 0$, in other words $\mathcal{F}_j$ are destabilizing sheaves.
- $H^0(q_C)(Z_j) = H^0(\mathcal{F}_j(n))$ where $q_C : V_n \otimes \mathcal{O}_C \to \mathcal{E}(n)$ is any surjective morphism of vector bundles and $Z_\bullet$ is the filtration induced by the one-parameter λ.

Now, we fix the morphism q_C and we set $\det \mathcal{E}(n) := \mathcal{L}_n$. Consider the morphism $S^m \bigwedge^r V_n \to H^0(C, \mathcal{L}_n^m)$ induced by q_c. The one-parameter subgroup λ acts on this morphism. Let $w(m)$ be the minimum among the sums of the weights of the elements of the basis $S^m \bigwedge^r B_n$ of $S^m \bigwedge^r V_n$ which induce, by using q_C, a basis of $H^0(C, \mathcal{L}_n^m)$. So $(C,\mathcal{E})$ is Hilbert-semistable for λ if and only if $w(m) \leq 0$ for $m \gg 0$ (see [TiB98, Recall 1.5]). It is enough to check the Hilbert-semistability for the one-parameter subgroups λ such that the associated chain of destabilizing sheaves is maximal. By hypothesis $\mathcal{E}_{C_1} = \bigoplus_{i=0}^k \mathcal{G}_i$, where $\mathcal{G}_0 = 0$ and $\mathcal{G}_i$ stable bundle of rank q and same slope of $\mathcal{E}_{C_1}$. Observe that $\mathcal{F}_j / \mathcal{E}_{C_2}(-\sum_i^N p_i) \cong \bigoplus_{i=0}^j \mathcal{G}_i$. Moreover if we set $\widetilde{Z}_j := \langle v_{\dim Z_{j-1}+1}, \ldots, v_{\dim Z_j}\rangle$ for $j = 1, \ldots, k$ and $\widetilde{Z}_0 := Z_0$ we have

$$\bigwedge^r V_n = \bigoplus_{\rho_0, \ldots, \rho_k \,|\, \sum \rho_j = r} W_{\rho_0, \ldots, \rho_k}, \quad \text{where} \quad W_{\rho_0, \ldots, \rho_k} := \bigwedge^{\rho_0} \widetilde{Z}_0 \otimes \ldots \otimes \bigwedge^{\rho_k} \widetilde{Z}_k.$$

An element of the basis $\bigwedge^r B_n$ contained in $W_{\rho_1, \ldots, \rho_k}$ has weight $w_{\rho_0, \ldots, \rho_k}(n) = \rho_0 \gamma_0(n) + \ldots + \rho_k \gamma_k(n)$. Where $\gamma_j(n)$ is the weight of an element of B_n inside $\widetilde{Z}_j$, i.e.

$$\gamma_j(n) = \sum_{i=0}^{k-1} \alpha_i \chi(\mathcal{F}_i(n)) - \sum_{i=j}^{k} \alpha_i \chi(\mathcal{E}(n)) \quad \text{where} \quad \alpha_k = 0$$

As in [Sch04, p. 186-187] the space of minimal weights which produces sections which do not vanish on C_1 is $W^1_{min} := W_{0,q,\ldots,q}$. The associated weight is

$$w_{1,min}(n) := \sum_{i=0}^{k-1} \alpha_i \left(\chi(\mathcal{F}_i(n)) r - \chi(\mathcal{E}(n)) iq\right).$$

Moreover a general section of W^1_{min} does not vanish at p_i. By [Sch04, Corollary 2.2.5] the space $S^m W^1_{min}$ generates $H^0\left(C_1, \left(\mathcal{L}_{n|C_1}\right)^m\right)$, so that the elements of $S^m \bigwedge^r B_n$ inside $S^m W^1_{min}$ will contribute with weight

$$K_1(n,m) := m(m(\deg \mathcal{E}_{C_1} + nr\omega_{C_1}) + 1 - g_{C_1}) w_{1,min}(n)$$

to a basis of $H^0\left(C_1, \left(\mathcal{L}_{n|C_1}\right)^m\right)$.

On the other hand the space of minimal weights which produces sections which do not vanish on C_2 is $W^2_{min} := W_{r,0,\ldots,0}$. The associated weight is

$$w_{2,min}(n) := \sum_{i=0}^{k-1} \alpha_i (\chi(\mathcal{F}_i(n)) - \chi(\mathcal{E}(n))) r$$

A general section of W^2_{min} vanishes at p_i with order r. By [Sch04, Corollary 2.2.5], the space $S^m W^2_{min}$ generates $H^0\left(C_1, \left(\mathcal{L}_{n|C_2}\left(-r\sum p_i\right)\right)^m\right)$, in particular the elements of $S^m \bigwedge^r B_n$ inside $S^m W^2_{min}$ will contribute with weight

$$K_2(n,m) := m(m(\deg \mathcal{E}_{C_2} - rN + nr\omega_{C_2}) + 1 - g_{C_2})w_{2,min}(n)$$

to a basis of $H^0\left(C_2, \left(\mathcal{L}_{n|C_2}\right)^m\right)$. It remains to find the elements in $S^m \bigwedge^r B_n$ which produce sections of minimal weight in $H^0\left(C_2, \left(\mathcal{L}_{n|C_2}\right)^m\right)$ vanishing with order less than mr on p_i for $i = 1, \ldots, N$. By a direct computation, we can see that the space of minimal weights which gives us sections with vanishing order $r-s$ at p_i such that $tq \leq s \leq (t+1)q$ (where $0 \leq t \leq k-1$) is

$$\mathbb{O}_{r-s} := W_{r-s,\underbrace{q\ldots,q}_{t},s-tq,0,\ldots,0}.$$

The associated weight is

$$w^{r-s}_{2,p_i}(n) := \sum_{i=0}^{k-1} \alpha_i \left(\chi(\mathcal{F}_i(n)) - \chi(\mathcal{E}(n))\right) r + \sum_{i=0}^{t} (s-iq)\alpha_i \chi(\mathcal{E}(n)).$$

For any $0 \leq \nu \leq mr-1$ and $1 \leq i \leq N$, we must find an element of minimal weight in $S^m \bigwedge^r B_n$ which produces a section in $H^0\left(C_2, \left(\mathcal{L}_{n|C_2}\right)^m\right)$ vanishing with order ν at p_i. Observe first that we can reduce to check it on the subspace

$$S^m\mathbb{O} = \bigoplus_{m_0,\ldots,m_r|\sum m_i = m} S^{m_0}\mathbb{O}_0 \otimes \ldots \otimes S^{m_r}\mathbb{O}_r.$$

A section in $S^{m_0}\mathbb{O}_0 \otimes \ldots \otimes S^{m_r}\mathbb{O}_r$ vanishes with order at least $\nu = m_1 + 2m_2 + \ldots + rm_r$ at p_i and we can find some with exactly that order. As explained in [Sch04, p. 191-192], an element of $S^m \bigwedge^r B_n$ of minimal weight, such that it produces a section of order ν at p_i, lies in

$$S^j\mathbb{O}_{t-1} \otimes S^{m-j}\mathbb{O}_t$$

where $\nu = mt - j$ and $1 \leq j \leq m$. So the mininum among the sums of the weights of the elements in $S^m \bigwedge^r B_n$ which give us a basis of

$$H^0\left(C_2, \left(\mathcal{L}_{n|C_2}\right)^m\right) / H^0\left(C_2, \left(\mathcal{L}_{n|C_2}(-r\sum p_i)\right)^m\right)$$

is

$$\begin{aligned}
D_2(n,m) &= N\left(m^2(w^1_{2,p_i}(n) + \ldots + w^r_{2,p_i}(n)) + \frac{m(m+1)}{2}(w^0_{2,p_i}(n) - w^r_{2,p_i}(n))\right) = \\
&= N\sum_{i=0}^{k-1}\alpha_i\left(m^2\left(\chi(\mathcal{F}_i(n))r^2 - \chi(\mathcal{E}(n))\left(r^2 - \frac{(r-iq-1)(r-iq)}{2}\right)\right) + \right. \\
&\quad \left. + \frac{m(m+1)}{2}(r-iq)\chi(\mathcal{E}(n))\right).
\end{aligned}$$

Then a basis for $H^0\left(C_2, \left(\mathcal{L}_{n|C_2}\right)^m\right)$ will have minimal weight $K_2(n,m) + D_2(n,m)$.

As in [Sch04, p.192-194] we obtain that $(C, \mathcal{E})$ will be Hilbert-semistable for λ if and only if exists n^* such that for $n \geq n^*$

$$P(n,m) = K_1(n,m) + K_2(n,m) + D_2(n,m) - mNw_{1,min}(n) \leq 0$$

as polynomial in m. A direct computation shows that $P(n,m) \leq 0$ as polynomial in m. So $(C, \mathcal{E})$ is H-semistable.
It remains to check that $(C, \mathcal{E})$ is not H-stable. It is enough to construct a one-parameter subgroup λ such that $(C, \mathcal{E})$ is strictly P-semistable respect to λ and $P(n,m) \equiv 0$ as polynomial in m. Fix a basis of $W_n := H^0(C, \mathcal{E}(n)_{C_2}(-\sum p_i))$ and complete to a basis $B_n := \{v_1, \ldots, v_{\dim V_n}\}$ of $V_n := H^0(C, \mathcal{E}(n))$. We define the one-paramenter subgroup λ of $SL(V_n)$ diagonalized by the basis B_n with weight vector

$$(\underbrace{\dim W_n - \dim V_n, \ldots, \dim W_n - \dim V_n}_{\dim W_n}, \underbrace{\dim W_n, \ldots, \dim W_n}_{\dim W_n - \dim V_n}).$$

A direct computation shows $P(n,m)\equiv 0$ (observe that it is the case when $\alpha_1=1$ and $\alpha_i=0$ for $2\le i\le k-1$ in the previous computation), which implies that $(C,\mathcal{E})$ is stricly H-semistable. $\square$

Proposition 3.1.12. *The generic point of an extremal boundary divisor is strictly P-semistable and strictly H-semistable.*

Proof. Fix $i\in\{0,\dots,\lfloor g/2\rfloor\}$ such that $\widetilde{\delta}_i^0$ is an extremal divisor. By Lemma 3.1.4 the generic point of the extremal boundary $\widetilde{\delta}_i^0$ is a curve C with two irreducible smooth components C_1 and C_2 of genus i and $g-i$ and a vector bundle $\mathcal{E}$ such that $\mathcal{E}_{C_1}$ is a stable vector bundle (or direct sum of stable vector bundles with same slope of $\mathcal{E}_{C_1}$ if $i=1$) and $\mathcal{E}_{C_2}$ is stable vector bundle. By Lemma 3.1.11 the generic point of $\widetilde{\delta}_i^0$ is strictly P-semistable and stricly H-semistable.
Suppose now that $i\neq g/2$ and consider the extremal boundary divisor $\widetilde{\delta}_i^r$. Take a point $(C,\mathcal{E})\in\widetilde{\delta}_i^0$ as above. Consider the destabilizing subsheaf $\mathcal{E}_{C_2}(-p)\subset\mathcal{E}$, where p is the unique node of C. Fix a basis of $W_n:=H^0(C,\mathcal{E}(n)_{C_2}(-p))$ and complete to a basis $\mathcal{V}:=\{v_1,\dots,v_{dimV_n}\}$ of $V_n=H^0(C,\mathcal{E}(n))$. We define the one-parameter subgroup λ of $SL(V_n)$ given with respect to the basis $\mathcal{V}$ by the weight vector

$$(\underbrace{\dim W_n-\dim V_n,\dots,\dim W_n-\dim V_n}_{\dim W_n},\underbrace{\dim W_n,\dots,\dim W_n}_{\dim W_n-\dim V_n}).$$

We have seen in the proof of Lemma 3.1.11 that the pair $(C,\mathcal{E})$ is strictly H-semistable respect to λ. In particular the limit respect to λ is strictly H-semistable. The limit will be a pair $(C',\mathcal{E}')$ such that C' is a semistable model for C and $\mathcal{E}'$ a properly balanced vector bundles such that the push-forward in the stabilization is the P-semistable sheaf

$$\mathcal{E}_{C_2}(-p)\oplus\mathcal{E}_{C_1}.$$

By Corollary 1.1.13(iii), $\mathcal{E}'_{C_1}\cong\mathcal{E}_{C_1}$ and $\mathcal{E}'_{C_2}\cong\mathcal{E}_{C_2}(-p)$. In particular, $\mathcal{E}$ has multidegree

$$\left(\deg\mathcal{E}'_{C_1},\deg\mathcal{E}'_R,\deg\mathcal{E}'_{C_2}\right)=\left(\deg_{C_1}\mathcal{E},r,\deg_{C_2}\mathcal{E}-r\right)=\left(d\frac{2i-1}{2g-2}-\frac{r}{2},r,d\frac{2(g-i)-1}{2g-2}-\frac{r}{2}\right)$$

Smoothing all nodal points on the rational chain R except the meeting point q between R and C_2, we obtain a generic point $(C'',\mathcal{E}'')$ in $\widetilde{\delta}_i^r$. It is H-semistable by the openess of the semistable locus. Let W''_n be a basis for $H^0\left(C'',\mathcal{E}''(n)_{C_1}(-q)\right)$ and complete to a basis $\mathcal{V}''_n$ of $H^0(C'',\mathcal{E}''(n))$. Let λ'' be the one parameter subgroup defined by the weight vector (with respect to the basis B)

$$(\underbrace{\dim W''_n-\dim V''_n,\dots,\dim W''_n-\dim V''_n}_{\dim W''_n},\underbrace{\dim W''_n,\dots,\dim W''_n}_{\dim W''_n-\dim V''_n}).$$

As in the proof of Lemma 3.1.11, a direct computation shows that $(C'',\mathcal{E}'')$ is strictly H-semistable respect to λ'', then also strictly P-semistable concluding the proof. $\square$

Using this, we obtain

Corollary 3.1.13. *We have an exact sequences of groups*

$$0\longrightarrow\bigoplus_{i=0,\dots,\lfloor g/2\rfloor}\oplus_{j\in J_i}\langle\mathcal{O}(\widetilde{\delta}_i^j)\rangle\longrightarrow Pic(\overline{\mathcal{V}ec}^{Hss}_{r,d,g})\longrightarrow Pic(\mathcal{V}ec^{ss}_{r,d,g})\longrightarrow 0$$

where the right map is the natural restriction and the left map is the natural inclusion. The same holds for the rigidification $\overline{\mathcal{V}}^{Hss}_{r,d,g}$.

3.2. **Picard group of $\mathcal{V}ec_{r,d,g}$.** In this section we will prove Theorem A. Note that the first three line bundles on the theorem are free generators for the Picard group of $\mathcal{J}ac_{d,g}$ (see Theorem 2.4.1) and the fourth line bundle restricted to $\mathcal{V}ec_{=\mathcal{L},C}$ freely generates its Picard group (see Theorem 2.5.1). By Lemma 3.1.5 together with Theorem 2.5.1(i) and Remark 2.5.2, we see that Theorem A(i) is equivalent to:

Theorem 3.2.1. *Assume that $g\ge 2$. For any smooth curve C and $\mathcal{L}$ line bundle of degree d over C we have an exact sequence.*

$$0\longrightarrow Pic(\mathcal{J}ac_{d,g})\longrightarrow Pic(\mathcal{V}ec^{ss}_{r,d,g})\longrightarrow Pic(\mathcal{V}ec^{ss}_{=\mathcal{L},C})\longrightarrow 0$$

For the rest of the subsection we will assume $g \geq 2$. Observe that the above theorem together with Lemma 3.1.5, Theorem 3.1.1 and Corollary 3.1.13 imply Theorem A(ii). Using Remark 3.1.10 together with Proposition 3.1.12, we deduce Theorem A(iii).

Let $\mathcal{J}ac^o_{d,g}$ (resp. $\mathcal{J}^o_{d,g}$) the open substack of $\mathcal{J}ac_{d,g}$ (resp. $\mathcal{J}_{d,g}$) which parametrizes the pairs $(C,\mathcal{L})$ such that $\mathrm{Aut}(C,\mathcal{L})=\mathbb{G}_m$. Note that $\mathcal{J}^o_{d,g}$ is a smooth irreducible variety, more precisely it is a moduli space of isomorphism classes of line bundle of degree d over a curve C satisfying the condition above.

Lemma 3.2.2. *There are isomorphisms*

$$Pic(\mathcal{J}ac_{d,g}) \cong Pic(\mathcal{J}ac^o_{d,g}), \quad Pic(\mathcal{J}_{d,g}) \cong Pic(\mathcal{J}^o_{d,g})$$

induced by the restriction maps.

Proof. We will prove the lemma for $\mathcal{J}^o_{d,g}$, the assertion for $\mathcal{J}ac^o_{d,g}$ will follow directly. We set $\mathcal{J}^*_{d,g} := \mathcal{J}_{d,g}\backslash\mathcal{J}^o_{d,g}$. By Theorem 2.1.4, it is enough to prove that the closed substack $\mathcal{J}^*_{d,g}$ has codimension ≥ 2. First we recall some facts about curves with non-trivial automorphisms: the closed locus $\mathcal{J}^{Aut}_{d,g}$ in $\mathcal{J}_{d,g}$ of curves with non-trivial automorphisms has codimension $g-2$ and it has a unique irreducible component $\mathcal{JH}_g$ of maximal dimension corresponding to the hyperelliptic curves (see [GV08, Remark 2.4]). Moreover in $\mathcal{JH}_{d,g}$ the closed locus $\mathcal{JH}^{extra}_{d,g}$ of hyperelliptic curves with extra-automorphisms has codimension $2g-3$ and it has a unique irreducible component of maximal dimension corresponding to the curves with an extra-involution (for details see [GV08, Proposition 2.1]).
By definition, $\mathcal{J}^*_{d,g} \subset \mathcal{J}^{Aut}_{d,g}$. By the facts above, it is enough to check the dimension of $\mathcal{J}^*_{d,g} \cap \mathcal{JH}_g \subset \mathcal{JH}_g$. With an abuse of notation, the stack $\mathcal{J}^*_{d,g} \cap \mathcal{JH}_{d,g}$, i.e. the locus of pairs $(C,\mathcal{L})$ such that C is hyperelliptic and $\mathrm{Aut}(C,\mathcal{L}) \neq \mathbb{G}_m$, will be called $\mathcal{J}^*_{d,g}$.
If $g \geq 4$, $\mathcal{JH}_{d,g}$ has codimension ≥ 2, then the lemma follows. If $g=3$ then $\mathcal{JH}_{d,3}$ is an irreducible divisor. It is enough to show that $\mathcal{J}^*_{d,3} \neq \mathcal{JH}_{d,3}$ and it is easy to check. If $g=2$, then all curves are hyperelliptic, $\dim \mathcal{J}_{d,2}=5$ and $\mathcal{JH}^{extra}_{d,g}$ has codimension 1. Consider the forgetful morphism $\mathcal{J}^*_{d,2} \to \mathcal{M}_2$. The fiber at C, when is non empty, is the closed subscheme of the Jacobian $J^d(C)$ where the action of $\mathrm{Aut}(C)$ is not free. If C is a curve without extra-automorphisms then the fiber has dimension 0. In particular if the open locus of such curves is dense in $\mathcal{J}^*_{d,2}$ then $\dim \mathcal{J}^*_{d,2} \leq \dim \mathcal{M}_2 = 3$ and the lemma follows. Otherwise, $\mathcal{J}^*_{d,2}$ can have an irreducible component of maximal dimension which maps in the divisor $\mathcal{H}^{extra}_2 \subset \mathcal{M}_2$ of curves with an extra-involution. In this case $\dim \mathcal{J}^*_{d,2} < \dim \mathcal{H}^{extra}_2 + \dim J^d(C) = 4$, which concludes the proof. ☐

We denote with $\mathcal{V}ec^o_{r,d,g}$ (resp. $\mathcal{V}^o_{r,d,g}$) the open substack of $\mathcal{V}ec^{ss}_{r,d,g}$ (resp. $\mathcal{V}^{ss}_{r,d,g}$) of pairs $(C,\mathcal{E})$ such that $\mathrm{Aut}(C,\det\mathcal{E})=\mathbb{G}_m$. By lemma above, Theorem 3.2.1 is equivalent to prove the exactness of

$$0 \longrightarrow \mathrm{Pic}(\mathcal{J}ac^o_{d,g}) \longrightarrow \mathrm{Pic}(\mathcal{V}ec^o_{r,d,g}) \longrightarrow \mathrm{Pic}(\mathscr{V}ec^{ss}_{=\mathcal{L},C}) \longrightarrow 0.$$

The morphism $det : \mathcal{V}ec^o_{r,d,g} \longrightarrow \mathcal{J}ac^o_{d,g}$ is a smooth morphism of Artin stacks. Let Λ be a line bundle over $\mathcal{V}ec^o_{r,d,g}$, which is obviously flat over $\mathcal{J}ac^o_{d,g}$ by flatness of the map det. The first step is to prove the following

Lemma 3.2.3. *Suppose that Λ is trivial over any geometric fiber. Then $det_*\Lambda$ is a line bundle on $\mathcal{J}ac^o_{d,g}$ and the natural map $det^*det_*\Lambda \longrightarrow \Lambda$ is an isomorphism.*

Proof. Consider the cartesian diagram

$$\begin{tikzcd} \mathcal{V}ec^o_{r,d,g} \arrow[d] & V_H \arrow[l] \arrow[d] \\ \mathcal{J}ac^o_{d,g} & H \arrow[l] \end{tikzcd}$$

where the bottom row is an atlas for $\mathcal{J}ac^o_{d,g}$. We can reduce to control the isomorphism locally on $V_H \to H$. Suppose that the following conditions hold

(i) H is an integral scheme,
(ii) the stack V_H has a good moduli scheme U_H,
(iii) U_H is proper over H with geometrically irreducible fibers.

Then, by Seesaw Principle (see Corollary B.10), we have the assertion. So it is enough to find an atlas H such that the conditions (i), (ii) and (iii) are satisfied.
We fix some notations: since the stack $\mathcal{V}ec^o_{r,d,g}$ is quasi-compact, there exists n big enough such that $\mathcal{V}ec^o_{r,d,g} \subset \overline{\mathcal{U}}_n = [H_n/GL(V_n)]$. So we can suppose d big enough such that $\mathcal{V}ec^o_{r,d,g} \subset \overline{\mathcal{U}}_0$. Let Q be the open subset of H_0 such that $\mathcal{V}ec^o_{r,d,g} = [Q/G]$, where $G := GL(V_0)$. Analogously, we set $\mathcal{J}ac^o_{d,g} = [H/\Gamma]$. Denote by $Z(\Gamma)$ (resp. $Z(G)$) the center of Γ (resp. of G) and set $\widetilde{G} = G/Z(G)$, $\widetilde{\Gamma} = \Gamma/Z(\Gamma)$. Note that $Z(G) \cong Z(\Gamma) \cong \mathbb{G}_m$. As usual we set $\mathcal{B}Z(\Gamma) := [\operatorname{Spec} k/Z(\Gamma)]$. Since $\mathcal{J}ac^o_{d,g}$ is integral then H is integral, satisfying the condition (i). We have the following cartesian diagrams

$$\begin{tikzcd}
& Q \arrow[d] & & Q \times_{\mathcal{J}^o_{d,g}} H \arrow[ll] \arrow[d] & Q \times_{\mathcal{J}ac^o_{d,g}} H \arrow[l, "\pi"'] \arrow[d] \\
& \mathcal{V}ec^o_{r,d,g} \arrow[dl] \arrow[ddd] & & {[Q \times_{\mathcal{J}^o_{d,g}} H/G]} \arrow[ll] \arrow[dl, "q"] \arrow[ddd] & {[Q \times_{\mathcal{J}ac^o_{d,g}} H/G] \cong V_H} \arrow[l, "p"'] \arrow[ddd] \\
\mathcal{V}^o_{r,d,g} \arrow[d] & & {[Q \times_{\mathcal{J}^o_{d,g}} H/\widetilde{G}]} \arrow[ll] \arrow[d] & & \\
U^o_{r,d,g} \arrow[dd] & & U_H \arrow[ll] \arrow[dd] & & \\
& \mathcal{J}ac^o_{d,g} \arrow[dl] & & H \times \mathcal{B}Z(\Gamma) \arrow[ll] \arrow[dl] & H \arrow[l] \arrow[dll, equal] \\
\mathcal{J}^o_{d,g} & & H \arrow[ll] & &
\end{tikzcd}$$

where $U^o_{r,d,g}$ is the open subscheme in $\overline{U}_{r,d,g}$ of pairs $(C, \mathcal{E})$ such that C is smooth and $\operatorname{Aut}(C, \det \mathcal{E}) = \mathbb{G}_m$. Note that U_H is proper over H, because $U^o_{r,d,g} \to \mathcal{J}^o_{d,g}$ is proper. In particular, the geometric fiber over a k-point of H which maps to $(C, \mathcal{L})$ in $\mathcal{J}^o_{d,g}$ is the irreducible projective variety $U_{\mathcal{L},C}$.
So it remains to prove that U_H is a good moduli space for V_H. Since V_H is a quotient stack, it is enough to show that U_H is a good G-quotient of $Q \times_{\mathcal{J}ac^o_{d,g}} H$. The good moduli morphisms are preserved by pull-backs [Alp13, Proposition 3.9], in particular U_H is a good G-quotient of $Q \times_{\mathcal{J}^o_{d,g}} H$. Consider the commutative diagram

$$\begin{tikzcd}
Q \times_{\mathcal{J}ac^o_{d,g}} H \arrow[r, "\pi"] \arrow[dr, "\beta"'] & Q \times_{\mathcal{J}^o_{d,g}} H \arrow[r] \arrow[d, "\alpha"] & U_H \arrow[dl] \\
& H &
\end{tikzcd}$$

<u>Claim:</u> the horizontal maps makes U_H a categorical G-quotient of $Q \times_{\mathcal{J}ac^0} H$.
Suppose that the claim holds. Then U_H is a good G-quotient also for $Q \times_{\mathcal{J}ac^o_{d,g}} H$, because the horizontal maps are affine (see [MFK94, 1.12]), and we have done.
It remains to prove the claim. The idea for this part comes from [Hof12, Section 2]. Since the map $Q \to \mathcal{J}^o_{d,g}$ is G-invariant then $Q \times_{\mathcal{J}^o_{d,g}} H \to H$ is G-invariant. In particular we can study the action of $Z(G)$ over the fibers of α. Fix a geometric point h on H and suppose that its image in $\mathcal{J}ac^o_{d,g}$ is the pair $(C, \mathcal{L})$. Then the fiber of β (resp. of α) over h is the fine moduli space of the triples $(\mathcal{E}, B, \phi)$ (resp. of the pairs $(\mathcal{E}, B)$), where $\mathcal{E}$ is a semistable vector bundle on C, B a basis of $H^0(C, \mathcal{E})$ and ϕ is an isomorphism between the line bundles $\det \mathcal{E}$ and $\mathcal{L}$. If $g \in Z(G)$ we have $g.(\mathcal{E}, B, \phi) = (\mathcal{E}, gB, \phi)$ and $g.(\mathcal{E}, B) = (\mathcal{E}, gB)$. Observe that the isomorphism $g.Id_{\mathcal{E}}$ gives us an isomorphism between the pairs $(\mathcal{E}, B)$ and $(\mathcal{E}, gB)$ and between the triples $(\mathcal{E}, B, \phi)$ and $(\mathcal{E}, gB, g^r\phi)$. So $g.(\mathcal{E}, B, \phi) = (\mathcal{E}, B, g^{-r}\phi)$ and $g.(\mathcal{E}, B) = (\mathcal{E}, B)$. On the other hand, $\pi : Q \times_{\mathcal{J}ac^o_{d,g}} H \to Q \times_{\mathcal{J}^o_{d,g}} H$ is a principal $Z(\Gamma)$-bundle and the group $Z(\Gamma)$ acts in the following way: if $\gamma \in Z(\Gamma)$ we have $\gamma.(\mathcal{E}, B, \phi) = (\mathcal{E}, B, \gamma\phi)$ and $\gamma.(\mathcal{E}, B) = (\mathcal{E}, B)$.
This implies that the groups $Z(G)/\mu_r$ (where μ_r is the finite algebraic group consisting of r-roots of unity)

and $Z(\Gamma)$ induce the same action on $Q\times_{\mathcal{J}ac^o_{d,g}} H$. Since $\pi: Q\times_{\mathcal{J}ac^o_{d,g}} H\to Q\times_{\mathcal{J}^o_{d,g}} H$ is a principal $Z(\Gamma)$-bundle, any G-invariant morphism from $Q\times_{\mathcal{J}ac^o_{d,g}} H$ to a scheme factorizes uniquely through $Q\times_{\mathcal{J}^o_{d,g}} H$ and so uniquely through U_H concluding the proof of the claim. ☐

The next lemma conclude the proof of Theorem 3.2.1.

Lemma 3.2.4. *Let Λ be a line bundle on $\mathcal{V}ec^o_{r,d,g}$. Then Λ is trivial on a geometric fiber of det if and only if Λ is trivial on any geometric fiber.*

Proof. Consider the determinant map $det: \mathcal{V}ec^o_{r,d,g}\to \mathcal{J}ac^o_{d,g}$. Let T be the set of points h (in the sense of [LMB00, Chap. 5]) in $\mathcal{J}ac^o_{d,g}$ such that the restriction $\Lambda_h:=\Lambda_{det^*h}$ is the trivial line bundle. By Theorem 2.5.1(iii), the inclusion

$$\mathrm{Pic}(U_{\mathcal{L},C})\hookrightarrow \mathrm{Pic}(\mathcal{V}ec^{ss}_{=\mathcal{L},C})\cong\mathbb{Z}$$

is of finite index. The variety $U_{\mathcal{L},C}$ is projective, in particular any non-trivial line bundle on it is ample or anti-ample. This implies that $\chi(\Lambda_h^n)$, as polynomial in the variable n, is constant if and only if Λ_h is trivial. So T is equal to the set of points h such that the polynomial $\chi(\Lambda_h^n)$ is constant. Consider the atlas defined in the proof of precedent lemma $H\to\mathcal{J}ac^o_{d,g}$. The line bundle Λ is flat over $\mathcal{J}ac^o_{d,g}$ so the function

$$\chi_n: H\to\mathbb{Z}:\ h=(C,\mathcal{L},B)\mapsto\chi(\Lambda_h^n)$$

is locally constant for any n, then constant because H is connected. Therefore, the condition $\chi_n=\chi_m$ for any n, $m\in\mathbb{Z}$ is either always satisfied or never satisfied, which concludes the proof. ☐

3.3. **Comparing the Picard groups of $\overline{\mathcal{V}ec}_{r,d,g}$ and $\overline{\mathcal{V}}_{r,d,g}$.** Assume that $g\geq 2$. Consider the rigidification map $\nu_{r,d}:\mathcal{V}ec_{r,d,g}\to\mathcal{V}_{r,d,g}$ and the sheaf of abelian groups $\mathbb{G}_m$. The Leray spectral sequence

$$H^p(\mathcal{V}_{r,d,g},R^q\nu_{r,d*}\mathbb{G}_m)\Rightarrow H^{p+q}(\mathcal{V}ec_{r,d,g},\mathbb{G}_m) \tag{3.3.1}$$

induces an exact sequence in low degrees

$$0\to H^1(\mathcal{V}_{r,d,g},\nu_{r,d*}\mathbb{G}_m)\to H^1(\mathcal{V}ec_{r,d,g},\mathbb{G}_m)\to H^0(\mathcal{V}_{r,d,g},R^1\nu_{r,d*}\mathbb{G}_m)\to H^2(\mathcal{V}_{r,d,g},\nu_{r,d*}\mathbb{G}_m).$$

We observe that $\nu_{r,d*}\mathbb{G}_m=\mathbb{G}_m$ and that the sheaf $R^1\nu_{r,d*}\mathbb{G}_m$ is the constant sheaf $H^1(\mathcal{B}\mathbb{G}_m,\mathbb{G}_m)\cong\mathrm{Pic}(\mathcal{B}\mathbb{G}_m)\cong\mathbb{Z}$. Via standard coycle computation we see that exact sequence becomes

$$0\longrightarrow\mathrm{Pic}(\mathcal{V}_{r,d,g})\longrightarrow\mathrm{Pic}(\mathcal{V}ec_{r,d,g})\xrightarrow{res}\mathbb{Z}\xrightarrow{obs}H^2(\mathcal{V}_{r,d,g},\mathbb{G}_m) \tag{3.3.2}$$

where res is the restriction on the fibers (it coincides with the weight map defined in [Hof07, Def. 4.1]), obs is the map which sends the identity to the $\mathbb{G}_m$-gerbe class $[\nu_{r,d}]\in H^2(\mathcal{V}_{r,d,g},\mathbb{G}_m)$ associated to $\nu_{r,d}:\mathcal{V}ec_{r,d,g}\to\mathcal{V}_{r,d,g}$ (see [Gir71, IV, §3.4-5]).

Lemma 3.3.1. *We have that:*

$$\begin{cases} res(\Lambda(1,0,0))=0,\\ res(\Lambda(0,0,1))=d+r(1-g),\\ res(\Lambda(0,1,0))=r(d+1-g),\\ res(\Lambda(1,1,0))=r(d-1+g).\end{cases}$$

Proof. Using the functoriality of the determinant of cohomology, we get that the fiber of $\Lambda(1,0,0)=d_\pi(\omega_\pi)$ over a point $(C,\mathcal{E})$ is canonically isomorphic to $\det H^0(C,\omega_C)\otimes\det^{-1}H^1(C,\omega_C)$. Since $\mathbb{G}_m$ acts trivially on $H^0(C,\omega_C)$ and on $H^1(C,\omega_C)$, we get that $res(\Lambda(1,0,0))=0$.
Similarly, the fiber of $\Lambda(0,0,1)$ over a point $(C,\mathcal{E})$ is canonically isomorphic to $\det H^0(C,\mathcal{E})\otimes\det^{-1}H^1(C,\mathcal{E})$. Since $\mathbb{G}_m$ acts with weight one on the vector spaces $H^0(C,\mathcal{E})$ and $H^1(C,\mathcal{E})$, Riemann-Roch gives that

$$res(\Lambda(0,0,1))=h^0(C,\mathcal{E})-h^1(C,\mathcal{E})=\chi(\mathcal{E})=d+r(1-g).$$

The fiber of $\Lambda(0,1,0)$ over a point $(C,\mathcal{E})$ is canonically isomorphic to $\det H^0(C,\det\mathcal{E})\otimes\det^{-1}H^1(C,\det\mathcal{E})$. Now $\mathbb{G}_m$ acts with weight r on the vector spaces $H^0(C,\det\mathcal{E})$ and $H^1(C,\det\mathcal{E})$, so that Riemann-Roch gives

$$res(\Lambda(0,1,0))=r\cdot h^0(C,\det\mathcal{E})-r\cdot h^1(C,\det\mathcal{E})=r\cdot\chi(\det\mathcal{E})=r(d+1-g).$$

Finally, the fiber of $\Lambda(1,1,0)$ over a point $(C,\mathcal{E})$ is canonically isomorphic to $\det H^0(C,\omega_C\otimes\det\mathcal{E})\otimes\det^{-1}H^1(C,\omega_C\otimes\det\mathcal{E})$. Since $\mathbb{G}_m$ acts with weight r on the vector spaces $H^0(C,\omega_C\otimes\det\mathcal{E})$ and $H^1(C,\omega_C\otimes\det\mathcal{E})$, Riemann-Roch gives that

$$res(\Lambda(1,1,0))=r\cdot h^0(C,\omega_C\otimes\det\mathcal{E})-r\cdot h^1(C,\omega_C\otimes\det\mathcal{E})=r\cdot\chi(\omega_C\otimes\det\mathcal{E})=r(d-1+g).$$

□

Combining the Lemma above with Theorem A(i) and the exact sequence (3.3.2), we obtain

Corollary 3.3.2.

(i) The image of $Pic(\mathcal{V}ec_{r,d,g})$ via the morphism res of (3.3.2) is the subgroup of $\mathbb{Z}$ generated by

$$n_{r,d}\cdot v_{r,d,g}=(d+r(1-g),r(d+1-g),r(d-1+g))\,.$$

(ii) The Picard group of $\mathcal{V}_{r,d,g}$ is (freely) generated by the line bundles $\Lambda(1,0,0)$, Ξ and Θ (when $g\geq 3$).

Now we are ready for

Proof of Theorem B. Corollary 3.3.2(ii) says that the Theorem B(i) is true for the stack $\mathcal{V}_{r,d,g}$. Using the Leray spectral sequence for the (semi)stable locus, we see that the Corollary 3.3.2 holds also for the stack $\mathcal{V}^{(s)s}_{r,d,g}$, concluding the proof of Theorem B(i).
By Corollary 3.1.9 and Theorem B(i), the Theorem B(ii) holds for $\overline{\mathcal{V}}_{r,d,g}$. Using the Lemma 3.1.5, the same is true for $\overline{\mathcal{V}}^{Pss}_{r,d,g}$. Finally by Corollary 3.1.13, Theorem B(ii) holds also for $\overline{\mathcal{V}}^{Hss}_{r,d,g}$.
The Theorem B(iii) follows using the previous parts together with Remark 3.1.10 and Proposition 3.1.12. □

Remark 3.3.3. Let $U_{r,d,g}$ be the coarse moduli space of aut-equivalence classes of semistable vector bundles on smooth curves. Suppose that $g\geq 3$. Kouvidakis in [Kou93] gives a description of the Picard group of the open subset $U^*_{r,d,g}$ of curves without non-trivial automorphisms. As observed in Section 3 of *loc. cit.*, such locus is locally factorial. Since the locus of strictly semistable vector bundles has codimension at least 2, we can restrict to study the open subset $U^\star_{r,d,g}\subset U_{r,d,g}$ of stable vector bundles on curves without non-trivial automorphisms. The good moduli morphism $\Psi_{r,d}:\mathcal{V}^{ss}_{r,d,g}\longrightarrow U_{r,d,g}$ is an isomorphism over $U^\star_{r,d,g}$. In other words, we have an isomorphism $\mathcal{V}^\star_{r,d,g}:=\Psi^{-1}_{r,d}(U^\star_{r,d,g})\cong U^\star_{r,d,g}$. Therefore, we get a natural surjective homomorphism

$$\psi:\mathrm{Pic}(\mathcal{V}^{ss}_{r,d,g})\cong\mathrm{Pic}(\mathcal{V}^{s}_{r,d,g})\twoheadrightarrow\mathrm{Pic}(\mathcal{V}^\star_{r,d,g})\cong\mathrm{Pic}(U^\star_{r,d,g})$$

where the first two homomorphisms are the restriction maps.
When $g\geq 4$ the codimension of $\mathcal{V}^s_{r,d,g}\backslash\mathcal{V}^\star_{r,d,g}$ is at least two (see [GV08, Remark 2.4]). Then the map ψ is an isomorphism by Theorem 2.1.4. If $g=3$, the locus $\mathcal{V}^{(s)s}_{r,d,3}\backslash\mathcal{V}^*_{r,d,3}$ is a divisor in $\mathcal{V}^{(s)s}_{r,d,3}$ (see [GV08, Remark 2.4]). More precisely is the pull-back of the hyperelliptic (irreducible) divisor in $\mathcal{M}_3$. As line bundle, it is isomorphic to Λ^9 in the Picard group of $\mathcal{M}_3$ (see [HM98, Chap. 3, Sec. E]). Therefore, by Theorem B(i), we get that $\mathrm{Pic}(U^*_{r,d,3})$ is the quotient of $\mathrm{Pic}(\mathcal{V}^{(s)s}_{r,d,3})$ by the relation $\Lambda(1,0,0)^9$.
In particular, (when $g\geq 3$) the line bundle $\Theta^s\otimes\Xi^t\otimes\Lambda(1,0,0)^u$, where $(s,t,u)\in\mathbb{Z}^3$, on $U^*_{r,d,g}$ has the same properties of the canonical line bundle $\mathscr{L}_{m,a}$ in [Kou93, Theorem 1], where $m=s\cdot\frac{v_{1,d,g}}{v_{r,d,g}}$ and $a=-s(\alpha+\beta)-t\cdot k_{1,d,g}$.

As explained in Section 1.3, $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$ admits a projective variety as good moduli space. This means, in particular, that the stacks $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$ and $\overline{\mathcal{V}}^{Hss}_{r,d,g}$ are of finite type and universally closed. Since any vector bundle contains the multiplication by scalars as automorphisms, $\overline{\mathcal{V}ec}^{Hss}_{r,d,g}$ is not separated. The next Proposition tell us exactly when the rigidification $\overline{\mathcal{V}}^{Hss}_{r,d,g}$ is separated.

Proposition 3.3.4. *The following conditions are equivalent:*

(i) $n_{r,d}\cdot v_{r,d,g}=1$, i.e. $n_{r,d}=1$ and $v_{r,d,g}=1$.
(ii) There exists a universal vector bundle on the universal curve of an open substack of $\overline{\mathcal{V}}_{r,d,g}$.
(iii) There exists a universal vector bundle on the universal curve of $\overline{\mathcal{V}}_{r,d,g}$.
(iv) The stack $\overline{\mathcal{V}}^{Hss}_{r,d,g}$ is proper.
(v) All H-semistable points are H-stable.

(vi) $\overline{\mathcal{V}}^{Hss}_{r,d,g}$ *is a Deligne-Mumford stack.*
(vii) The stack $\overline{\mathcal{V}}^{Pss}_{r,d,g}$ *is proper.*
(viii) All P-semistable points are P-stable.
(ix) $\overline{\mathcal{V}}^{Pss}_{r,d,g}$ *is a Deligne-Mumford stack.*

Proof. The strategy of the proof is the following

$$\begin{tikzcd}
(iii) \arrow[d, Rightarrow] & (i) \arrow[l, Rightarrow] \arrow[d, Leftrightarrow] & (ix) \arrow[r, Rightarrow] & (v) \arrow[r, Leftrightarrow] \arrow[d, Leftrightarrow] \arrow[ll, bend right, Rightarrow] & (iv) \\
(ii) \arrow[ur, Rightarrow] & (viii) \arrow[r, Rightarrow] \arrow[ur, Rightarrow] & (vii) \arrow[ur, Rightarrow] & (vi) &
\end{tikzcd}$$

$(i) \Rightarrow (iii)$. By Corollary 3.3.2(i) and the exact sequence (3.3.2), any line bundle on $\overline{\mathcal{V}ec}_{r,d,g}$ must have weight equal to $c \cdot n_{r,d} \cdot v_{r,d,g}$, where $c \in \mathbb{Z}$. In particular the condition (i) is equivalent to have a line bundle $\mathcal{L}$ of weight 1 on $\overline{\mathcal{V}ec}_{r,d,g}$. Let $\left(\overline{\pi} : \overline{\mathcal{V}ec}_{r,d,g,1} \to \overline{\mathcal{V}ec}_{r,d,g}, \mathcal{E}\right)$ be the universal pair, we see easily that $\mathcal{E} \otimes \overline{\pi}^* \mathcal{L}^{-1}$ descends to a vector bundle on $\overline{\mathcal{V}}_{r,d,g}$ with the universal property.
$(iii) \Rightarrow (ii)$ Obvious.
$(ii) \Rightarrow (i)$ Suppose that there exists a universal pair $(\mathcal{S}_1 \to \mathcal{S}, \mathcal{F})$ on some open substack $\mathcal{S}$ of $\overline{\mathcal{V}}_{r,d,g}$. We can suppose that all the points $(C, \mathcal{E})$ in $\mathcal{S}$ are such that $\mathrm{Aut}(C, \mathcal{E}) = \mathbb{G}_m$. Let $\nu_{r,d} : \mathcal{T} := \nu_{r,d}^{-1}\mathcal{S} \to \mathcal{S}$ be the restriction of the rigidification map and $(\overline{\pi} : \mathcal{T}_1 \to \mathcal{T}, \mathcal{E})$ the universal pair on $\mathcal{T} \subset \overline{\mathcal{V}ec}_{r,d,g}$. Then

$$\overline{\pi}_* \left(Hom\left(\nu_{r,d}^* \mathcal{F}, \mathcal{E}\right)\right)$$

is a line bundle of weight 1 on $\mathcal{T}$ and, by smoothness of $\overline{\mathcal{V}ec}_{r,d,g}$, we can extend it to a line bundle of weight 1 on $\overline{\mathcal{V}ec}_{r,d,g}$.
$(iv) \iff (v)$. If all H-semistable points are H-stable, then by [MFK94, Corollary 2.5] the action of $GL(V_n)$ on the H-semistable locus of H_n is proper, i.e. the morphism $PGL \times H_n^{Hss} \to H_n^{Hss} \times H_n^{Hss} : (A, h) \mapsto (h, A.h)$ is proper (for n big enough). Consider the cartesian diagram

$$\begin{tikzcd}
PGL \times H_n^{Hss} \arrow[r] \arrow[d] & H_n^{Hss} \times H_n^{Hss} \arrow[d] \\
\overline{\mathcal{V}}^{Hss}_{r,d,g} \arrow[r] & \overline{\mathcal{V}}^{Hss}_{r,d,g} \times \overline{\mathcal{V}}^{Hss}_{r,d,g}
\end{tikzcd}$$

this implies that the diagonal is proper, i.e. the stack is separated. We have already seen that it is always universally closed and of finite type, so it is proper. Conversely, if the diagonal is proper the automorphism group of any point must be finite, in particular there are no strictly H-semistable points.
$(v) \iff (vi)$. By [LMB00, Theorem 8.1], $\overline{\mathcal{V}}^{Hss}_{r,d,g}$ is Deligne-Mumford if and only if the diagonal is unramified, which is also equivalent to the fact that the automorphism group of any point is a finite group (because we are working in characteristic 0). As before, this happens if and only if all semistable points are stable.
$(v), (viii) \Rightarrow (i)$. It is known that, on smooth curves, $n_{r,d} = 1$ if and only if all semistable vector bundles are stable. So we can suppose that $n_{r,d} = 1$, so that $v_{r,d,g} = (2g-2, d+1-g, d+r(1-g)) = (2g-2, d+r(1-g))$. If $v_{r,d,g} \neq 1$ we have $k_{r,d,g} < 2g-2$, we can construct a nodal curve C of genus g, composed by two irreducible smooth curves C_1 and C_2 meeting at N points, such that $\omega_{C_1} = k_{r,d,g}$. In particular $(d_1, d_2) := (d\frac{\omega_{C_1}}{\omega_C} - N\frac{r}{2}, d\frac{\omega_{C_2}}{\omega_C} + N\frac{r}{2})$ are integers. So we can construct a vector bundle $\mathcal{E}$ on C with multidegree (d_1, d_2) and rank r satisfying the hypothesis of Lemma 3.1.11. This implies that the pair $(C, \mathcal{E})$ must be strictly P-semistable and strictly H-semistable.
$(i) \Rightarrow (viii)$. Suppose that there exists a point $(C, \mathcal{E})$ in $\overline{\mathcal{V}}_{r,d,g}$ such that $(C^{st}, \pi_*\mathcal{E})$ is strictly P-semistable. If C is smooth then $n_{r,d} \neq 1$ and we have done. Suppose that $n_{r,d} = 1$ and C singular. By hypothesis there exists a destabilizing subsheaf $\mathcal{F} \subset \pi_*\mathcal{E}$, such that

$$\frac{\chi(\mathcal{F})}{\sum s_i \omega_{C_i}} = \frac{\chi(\mathcal{E})}{r\omega_C}.$$

The equality can exist if and only if $(\chi(\mathcal{E}), r\omega_C) = (d + r(1-g), r(2g-2)) \neq 1$. We have supposed that d and r are coprime, so $(d+r(1-g), r(2g-2)) = (d+r(1-g), 2g-2) = (2g-2, d+1-g, d+r(1-g)) = v_{r,d,g}$,

which concludes the proof.
$(viii) \Rightarrow (vii),(ix)$. By hypothesis $\overline{\mathcal{V}}^{Pss}_{r,d,g} = \overline{\mathcal{V}}^{Ps}_{r,d,g} = \overline{\mathcal{V}}^{Hss}_{r,d,g} = \overline{\mathcal{V}}^{Hs}_{r,d,g}$, so (vi) and $(viii)$ hold by what proved above.
$(vii),(ix) \Rightarrow (v)$. Suppose that (v) does not hold, then there exists a strictly H-semistable point with automorphism group of positive dimension. Thus $\overline{\mathcal{V}}^{Hss}_{r,d,g}$, and in particular $\overline{\mathcal{V}}^{Pss}_{r,d,g}$, cannot be neither proper nor Deligne-Mumford. □

Appendix A. Genus Two case.

In this appendix we will extend the Theorems A and B to the genus two case. The main results are the following

Theorem A.1. *Suppose that $r \geq 2$.*

(i) The Picard groups of $\mathcal{V}ec_{r,d,2}$ and $\mathcal{V}ec^{ss}_{r,d,2}$ are generated by $\Lambda(1,0,0)$, $\Lambda(1,1,0)$, $\Lambda(0,1,0)$ and $\Lambda(0,0,1)$ with the unique relation

$$\Lambda(1,0,0)^{10} = \mathcal{O}. \tag{A.0.1}$$

(ii) The Picard groups of $\overline{\mathcal{V}ec}_{r,d,2}$ and $\overline{\mathcal{V}ec}^{Pss}_{r,d,2}$ are generated by $\Lambda(1,0,0)$, $\Lambda(1,1,0)$, $\Lambda(0,1,0)$, $\Lambda(0,0,1)$ and the boundary line bundles with the unique relation

$$\Lambda(1,0,0)^{10} = \mathcal{O}\left(\widetilde{\delta}_0 + 2\sum_{j\in J_1} \widetilde{\delta}_1^j\right) \tag{A.0.2}$$

Let $v_{r,d,2}$ and $n_{r,d}$ be the numbers defined in the Notations 0.0.1. Let α and β be (not necessarily unique) integers such that $\alpha(d-1)+\beta(d+1) = -\frac{1}{n_{r,d}}\cdot\frac{v_{1,d,2}}{v_{r,d,2}}(d-r)$. We set

$$\Xi := \Lambda(0,1,0)^{\frac{d+1}{v_{1,d,2}}} \otimes \Lambda(1,1,0)^{-\frac{d-1}{v_{1,d,2}}}, \quad \Theta := \Lambda(0,0,1)^{\frac{r}{n_{r,d}}\cdot\frac{v_{1,d,2}}{v_{r,d,2}}} \otimes \Lambda(0,1,0)^{\alpha} \otimes \Lambda(1,1,0)^{\beta}.$$

Theorem A.2. *Suppose that $r \geq 2$.*

(i) The Picard groups of $\mathcal{V}_{r,d,g}$ and $\mathcal{V}^{ss}_{r,d,g}$ are generated by $\Lambda(1,0,0)$, Ξ and Θ, with the unique relation (A.0.1).

(ii) The Picard groups of $\overline{\mathcal{V}}_{r,d,2}$ and $\overline{\mathcal{V}}^{Pss}_{r,d,2}$ are generated by $\Lambda(1,0,0)$, Ξ, Θ and the boundary line bundles with the unique relation (A.0.2).

Unfortunately, at the moment we can not say if the Theorems A and B hold also for the other open substacks in the assertions.

Remark A.3. Observe that, using Proposition 3.1.2, we can prove that Lemma 3.1.5 holds also in genus two case. In particular, by Theorem 2.1.4, we have that $\mathrm{Pic}(\overline{\mathcal{V}ec}_{r,d,2}) \cong \mathrm{Pic}(\overline{\mathcal{V}ec}^{Pss}_{r,d,2}) \cong \mathrm{Pic}(\overline{\mathcal{U}}_n)$ and $\mathrm{Pic}(\mathcal{V}ec_{r,d,2}) \cong \mathrm{Pic}(\mathcal{V}ec^{ss}_{r,d,2}) \cong \mathrm{Pic}(\mathcal{U}_n)$ for n big enough.

We have analogous isomorphisms for the rigidified moduli stacks.

*Proof of Theorem*A.1(i) and A.2(i). By the precedent observation, it is enough to prove the theorems for the semistable locus. Let $(C,\mathcal{L})$ be a k-point of $\mathcal{J}ac_{d,2}$. We recall that Theorem 3.2.1 says that the complex of groups

$$0 \longrightarrow \mathrm{Pic}(\mathcal{J}ac_{d,2}) \longrightarrow \mathrm{Pic}(\mathcal{V}ec^{ss}_{r,d,2}) \longrightarrow \mathrm{Pic}(\mathcal{V}ec^{ss}_{=\mathcal{L},C}) \longrightarrow 0$$

is exact. By Theorem 2.5.1, the cokernel is freely generated by the restriction of the line bundle $\Lambda(0,0,1)$ on the fiber $\mathcal{V}ec^{ss}_{=\mathcal{L},C}$. In particular the Picard groups of $\mathcal{V}ec_{r,d,2}$ and $\mathcal{V}ec^{ss}_{r,d,2}$ decomposes in the following way

$$\mathrm{Pic}(\mathcal{J}ac_{d,2}) \oplus \langle\Lambda(0,0,1)\rangle.$$

By Theorem 2.4, Theorem A.1(i) follows. By Corollary 3.3.2, Theorem A.2(i) also holds. □

Now we are going to prove the Theorems A.1(ii) and A.2(ii). First of all, by Theorems A.1(i) and 2.1.4, we know that the Picard group of $\overline{\mathcal{V}ec}_{r,d,2}$ is generated by $\Lambda(1,0,0)$, $\Lambda(1,1,0)$, $\Lambda(0,1,0)$, $\Lambda(0,0,1)$ and the boundary line bundles. Consider the forgetful map $\overline{\phi}_{r,d} : \overline{\mathcal{V}ec}_{r,d,2} \to \overline{\mathcal{M}}_2$. By Theorem 2.3.1, the Picard group of $\overline{\mathcal{M}}_2$ is generated by the line bundles δ_0, δ_1 and the Hodge line bundle Λ, with the unique relation

$\Lambda^{10} = \mathcal{O}(\delta_0 + 2\delta_1)$. By pull-back along $\overline{\phi}_{r,d}$ we obtain the relation (A.0.2). So for proving Theorem A.1(ii), it remains to show that we do not have other relations on $\mathrm{Pic}(\overline{\mathcal{V}ec}_{r,d,2})$.

Suppose there exists another relation, i.e.

$$\Lambda(1,0,0)^a \otimes \Lambda(1,1,0)^b \otimes \Lambda(0,1,0)^c \otimes \Lambda(0,0,1)^d \otimes \mathcal{O}\left(e_0\widetilde{\delta}_0 + \sum_{j\in J_1} e_1^j \widetilde{\delta}_1^j\right) = \mathcal{O} \tag{A.0.3}$$

where $a, b, c, d, e_o, e_1^j \in \mathbb{Z}$. By Theorem A.1(i), the integers b, c, d must be 0 and a must be a multiple of 10. We set $a = 10t$. Combining the equalities (A.0.2) and (A.0.3) we obtain:

$$\mathcal{O}\left((e_0 - t)\widetilde{\delta}_0 + \sum_{j\in J_1} (e_1^j - 2t)\widetilde{\delta}_1^j\right) = \mathcal{O} \tag{A.0.4}$$

where the integers $(e_0 - t), (e_1^j - 2t)$ cannot be all equal to 0, because we have assumed that the two relations are independent. In other words the existence of two independent relations is equivalent to show that does not exist any relation among the boundary line bundles. We will show this arguing as in §3.1. Observe that, arguing in the same way, we can arrive at same conclusions for the rigidified moduli stack $\overline{\mathcal{V}}_{r,d,2}$.

The Family $\widetilde{G}$.
Consider a double covering Y' of $\mathbb{P}^2$ ramified along a smooth sextic D. Consider on it a general pencil of hyperplane sections. By blowing up Y' at the base locus of the pencil we obtain a family $\varphi : Y \to \mathbb{P}^1$ of irreducible stable curves of genus two with at most one node. Moreover the two exceptional divisors E_1, $E_2 \subset Y$ are sections of φ trough the smooth locus of φ. The vector bundle $\mathcal{E} := \mathcal{O}_Y(dE_1) \oplus \mathcal{O}_Y^{r-1}$ is properly balanced of relative degree d. We call G (resp. $\widetilde{G}$) the family of curves $\varphi : Y \to \mathbb{P}^1$ (resp. the family φ with the vector bundle $\mathcal{E}$). We claim that

$$\begin{cases} \deg_{\widetilde{G}} \mathcal{O}(\widetilde{\delta}_0) = 30, \\ \deg_{\widetilde{G}} \mathcal{O}(\widetilde{\delta}_1^j) = 0 \qquad \text{for any } j \in J_1. \end{cases}$$

The second result comes from the fact that all fibers of φ are irreducible. We recall that, as §3.1: $\deg_{\widetilde{G}} \mathcal{O}(\widetilde{\delta}_0) = \deg_G \mathcal{O}(\delta_0)$. So our problem is reduced to check the degree on $\overline{\mathcal{M}}_2$. Observe also that Y is smooth and the generic fiber of φ is a smooth curve. Since any fiber of $\varphi : Y \to \mathbb{P}^{-1}$ can have at most one node and the total space Y is smooth, by [AC87, Lemma 1], $\deg_{\widetilde{G}} \mathcal{O}(\widetilde{\delta}_0)$ is equal to the number of singular fibers of φ. We can count them using the morphism $\varphi_D : D \to \mathbb{P}^1$, induced by the pencil restricted to the sextic D. By the generality of the pencil, we can assume that over any point of $\mathbb{P}^1$ there is at most one ramification point and that its ramification index at this point is 2. So $\deg_{\widetilde{G}} \mathcal{O}(\widetilde{\delta}_0)$ is equal to the degree of the ramification divisor in D. Using the Riemann-Hurwitz formula for the degree six morphism φ_D we obtain the first equality.

The Families $\widetilde{G}_1^j$.
Consider a general pencil of cubics in $\mathbb{P}^2$. Blowing up the nine base points of the pencil, we obtain a family of irreducible stable elliptic curves $\phi : X \to \mathbb{P}^1$. The nine exceptional divisors $E_1, \ldots, E_9 \subset X$ are sections of ϕ trough the smooth locus of ϕ. The family will have twelve singular fibers consisting of irreducible nodal elliptic curves. Fix a smooth elliptic curve Γ and a point $\gamma \in \Gamma$. We construct a surface Y by setting

$$Y = \left(X \coprod (\Gamma \times \mathbb{P}^1)\right) / (E_1 \sim \{\gamma\} \times \Gamma)$$

We get a family $f : X \to \mathbb{P}^1$ of stable curves of genus two. The general fiber is as in Figure 8 where C is a smooth elliptic curve. While the twelve special fibers are as in Figure 9 where C is a nodal irreducible elliptic curve. Choose a vector bundle M^j of degree $\lceil \frac{d-r}{2} \rceil + j$ on Γ, pull it back to $\Gamma \times \mathbb{P}^1$ and call it again M^j. Since M^j is trivial on $\{\gamma\} \times \mathbb{P}^1$, we can glue it with the vector bundle

$$\mathcal{O}_X\left(\left(\left\lfloor \frac{d+r}{2} \right\rfloor - j\right) E_2\right) \oplus \mathcal{O}_X^{r-1}$$

on X obtaining a vector bundle $\mathcal{E}^j$ on $f : X \to \mathbb{P}^1$ of relative rank r and degree d. The next lemma follows easily

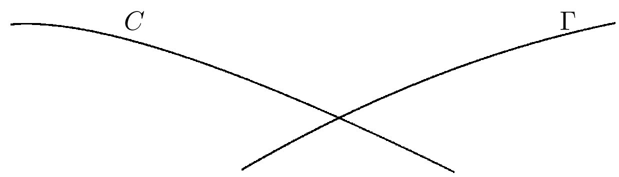

FIGURE 8. The general fiber of $f : X \to \mathbb{P}^1$.

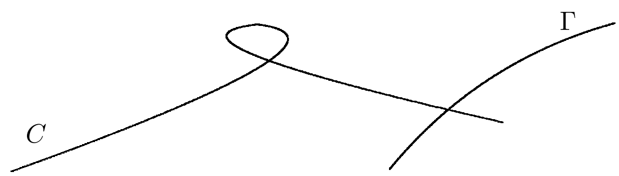

FIGURE 9. The special fibers of $f : X \to \mathbb{P}^1$.

Lemma A.4. *The vector bundle $\mathcal{E}^j$ is a properly balanced for $j \in J_1 = \{0, \dots, \lfloor r/2 \rfloor\}$.*

We call G_1 (resp. $\widetilde{G}_1^j$) the family of curves $f : X \to \mathbb{P}^1$ (resp. the family f with the vector bundle $\mathcal{E}^j$). Moreover $\widetilde{G}_1^j$ does not intersect $\widetilde{\delta}_1^k$ for $j \neq k$. In particular $\deg_{\widetilde{G}_1^j} \mathcal{O}(\widetilde{\delta}_1^j) = \deg_{G_1} \mathcal{O}(\delta_1)$. By [AC87, Lemma 1], the divisor $\mathcal{O}(\delta_1)$ restricted to the family G_1 is isomorphic to the tensor product between the normal bundle of E_1 in X and the normal bundle of $\gamma \times \mathbb{P}^1$ in $\Gamma \times \mathbb{P}^1$, i.e. $N_{E_1/X} \otimes N_{\{\gamma\} \times \mathbb{P}^1/\Gamma \times \mathbb{P}^1}$. The first factor has degree -1, while the second is trivial. Putting all together, we get

$$\begin{cases} \deg_{\widetilde{G}_1^k} \mathcal{O}(\widetilde{\delta}_1^k) = -1, \\ \deg_{\widetilde{G}_1^k} \mathcal{O}(\widetilde{\delta}_1^j) = 0 \qquad \text{if } j \neq k. \end{cases}$$

Now we can finally conclude the proof of Theorems A.1 and A.2.

Proof of Theorem A.1(ii) and A.2(ii). Suppose there exists a non-trivial relation $\mathcal{O}(a_0 \widetilde{\delta}_0 + \sum a_1^j \widetilde{\delta}_1^j) = \mathcal{O}$. If we restrict this equality on $\widetilde{G}$ we have $a_0 = 0$. Pulling back to G_1^j we deduce $a_1^j = 0$ for any $j \in J_1$. This concludes the proof of A.1(ii). Repeating the same arguments for the rigidified moduli stack $\overline{\mathcal{V}}_{r,d,2}$ we prove Theorem A.2(ii). □

APPENDIX B. BASE CHANGE COHOMOLOGY FOR STACKS ADMITTING A GOOD MODULI SPACE.

We will prove that the classical results of base change cohomology for proper schemes continue to hold again (not necessarily proper) stacks, which admit a proper scheme as good moduli space (in the sense of Alper). The propositions and proofs are essentially equal to ones in [Bro12, Appendix A], but we rewrite them, because our hypothesis are weaker.
In this section, $\mathcal{X}$ will be an Artin stack of finite type over a scheme S, and a sheaf $\mathcal{F}$ will be a sheaf for the site lisse-étale defined in [LMB00, Sec. 12] (see also [Bro, Appendix A]). Recall first the definition of good moduli space.

Definition B.1. [Alp13, def 4.1] Let S be a scheme, $\mathcal{X}$ be an Artin Stack over S and X an algebraic space over S . We call an S-morphism $\pi : \mathcal{X} \to X$ a good moduli space if

- π is quasi-compact,
- π_* is exact,
- The structural morphism $\mathcal{O}_X \to \pi_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism.

Remark B.2. Let $\mathcal{X}$ be a quotient stack of a quasi-compact k-scheme X by a smooth affine linearly reductive group scheme G. Suppose that $\mathcal{L}$ is a G-linearization on X. By[Alp13, Theorem 13.6 and Remark 13.7], the GIT good quotient $X^{ss}_{\mathcal{L}} /\!\!/_{\mathcal{L}} G$ is a good moduli space for the open substack $[X^{ss}_{\mathcal{L}}/G]$.
Conversely, suppose that there exists an open $U \subset X$ such that the open substack $[U/G]$ admits a good moduli space Y. By [Alp13, Theorem 11.14], there exists a G-linearized line bundle $\mathcal{L}$ over X such that U is contained in $X^{ss}_{\mathcal{L}}$, $[U/G]$ is saturated respect to the morphism $[X^{ss}_{\mathcal{L}}/G] \to X^{ss}_{\mathcal{L}} /\!\!/_{\mathcal{L}} G$ and Y is the GIT good quotient $U /\!\!/_{\mathcal{L}} G$.

Before stating the main result of this Appendix, we need to recall the following

Lemma B.3. ([Mum70, Lemma 1, II], *see also* [Bro12, Lemma 4.1.3]).

(i) Let A be a ring and let $C^\bullet$ be a complex of A-modules such that $C^p \neq 0$ only if $0 \leq p \leq n$. Then there exists a complex $K^\bullet$ of A-modules such that $K^p \neq 0$ only if $0 \leq p \leq n$ and K^p is free if $1 \leq p \leq n$, and a quasi-isomorphism of complexes $K^\bullet \to C^\bullet$. Moreover, if the C^p are flat, then K^0 will be A-flat too.

(ii) If A is noetherian and if the $H^i(C^\bullet)$ are finitely generated A-modules, then the K^p's can be chosen to be finitely generated.

Proposition B.4. *Let $\mathcal{X}$ be a quasi-compact Artin stack over an affine scheme (resp. noetherian affine scheme) $S = Spec(A)$. Let $\pi : \mathcal{X} \to X$ be a good moduli space with X separated (resp. proper) scheme over S. Let $\mathcal{F}$ be a quasi-coherent (resp. coherent) sheaf on $\mathcal{X}$ that is flat over S. Then there is a complex of flat A-modules (resp. of finite type)*

$$0 \longrightarrow M^0 \longrightarrow M^1 \longrightarrow \ldots \longrightarrow M^n \longrightarrow 0$$

with M^i free over A for $1 \leq i \leq n$, and isomorphisms

$$H^i(M^\bullet \otimes_A A') \longrightarrow H^i(\mathcal{X} \otimes_A A', \mathcal{F} \otimes_A A')$$

functorial in the A-algebra A'.

Proof. We consider the Cech complex $C^\bullet(\mathcal{U}, \pi_*\mathcal{F})$ associated to an affine covering $\mathcal{U} = (U_i)_{i \in I}$ of X. It is a finite complex of flat (by [Alp13, Theorem 4.16(ix)]) A-modules. Moreover, since X is separated, then we have $H^i(C^\bullet(\mathcal{U}, \pi_*\mathcal{F})) \cong H^i(X, \pi_*\mathcal{F})$. If A' is an A-algebra, then the covering $\mathcal{U} \otimes_A A'$ is still affine by S-separateness of X. This implies that

$$H^i(C^\bullet(\mathcal{U}, \pi_*\mathcal{F})) \otimes_A A' \cong H^i(X \otimes_A A', (\pi_*\mathcal{F}) \otimes_A A').$$

By [Alp13, Proposition 4.5], we have

$$H^i(X \otimes_A A', (\pi_*\mathcal{F}) \otimes_A A') \cong H^i(X \otimes_A A', \pi_*(\mathcal{F} \otimes_A A')).$$

Since π_* is exact, the Leray-spectral sequence $H^i(X \otimes_A A', R^j\pi_*(\mathcal{F} \otimes_A A')) \Rightarrow H^{i+j}(\mathcal{X} \otimes_A A', (F \otimes_A A'))$ (see [Bro, Theorem. A.1.6.4]) degenerates in the isomorphisms $H^i(X \otimes_A A', \pi_*(\mathcal{F} \otimes_A A')) \cong H^i(\mathcal{X} \otimes_A A', \mathcal{F} \otimes_A A')$. Putting all together:

$$H^i(C^\bullet(\mathcal{U}, \pi_*\mathcal{F})) \otimes_A A' \cong H^i(\mathcal{X} \otimes_A A', \mathcal{F} \otimes_A A').$$

It can be check that such isomorphisms are functorial in the A-algebra A'. Observe that if $\mathcal{F}$ is coherent then also $\pi_*\mathcal{F}$ is coherent (see [Alp13, Theorem 4.16(x)]). So if X is proper, then the modules $H^i(\mathcal{X}, \mathcal{F})$ are finitely generated. In particular, the cohomology modules of the complex $C^\bullet(\mathcal{U}, \pi_*\mathcal{F})$ are finitely generated. We can use the precedent lemma for conclude the proof. □

From the above results, we deduce several useful Corollaries.

Corollary B.5. *Let S be a scheme and let $q : \mathcal{X} \to S$ be a quasi-compact Artin Stack with an S-separated scheme X as good moduli space. Let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathcal{X}$ flat over S. If all sheaves $R^iq_*\mathcal{F}$ are flat over S then $\mathcal{F}$ is cohomologically flat.*

Proof. See [Bro12, Corollary 2.6] □

The proofs of next results are the same of [Mum70, II.5].

Corollary B.6. *Let $\mathcal{X} \to X$ be a good moduli space over a scheme S, X proper scheme over S and $\mathcal{F}$ coherent sheaf over $\mathcal{X}$ flat over S. Then we have:*

(i) for any $p \geq 0$ the function $S \to \mathbb{Z}$ defined by $s \mapsto dim_{k(s)}H^i(\mathcal{X}_s, \mathcal{F}_s)$ is upper semicontinuous on S.

(ii) The function $S \to \mathbb{Z}$ defined by $s \mapsto \chi(\mathcal{F}_s)$ is locally constant.

Corollary B.7. *Let $\mathcal{X} \to X$ be a good moduli space over an integral scheme S, X proper scheme over S and $\mathcal{F}$ coherent sheaf over $\mathcal{X}$ flat over S. The following conditions are equivalent*

(i) $s \mapsto dim_{k(s)} H^i(\mathcal{X}_s, \mathcal{F}_s)$ is a constant function,

(ii) $R^i q_*(\mathcal{F})$ is locally free sheaf on S and for any $s \in S$ the map

$$R^i q_*(\mathcal{F}) \otimes k(s) \to H^i(\mathcal{X}_s, \mathcal{F}_s)$$

is an isomorphism.

If these conditions are satisfied, then we have an isomorphism

$$R^{i-1} q_*(\mathcal{F}) \otimes k(s) \to H^{i-1}(\mathcal{X}_s, \mathcal{F}_s)$$

Corollary B.8. *Let $\mathcal{X} \to X$ be a good moduli space over a scheme S, X proper scheme over S and $\mathcal{F}$ coherent sheaf over $\mathcal{X}$ flat over S. Assume for some i that $H^i(\mathcal{X}_s, \mathcal{F}_y) = (0)$ for any $s \in S$. Then the natural map*

$$R^{i-1} q_*(\mathcal{F}) \otimes_{\mathcal{O}_S} k(s) \to H^{i-1}(\mathcal{X}_s, \mathcal{F}_y)$$

is an isomorphism for any $s \in S$.

Corollary B.9. *Let $\mathcal{X} \to X$ be a good moduli space over a scheme S, X proper scheme and $\mathcal{F}$ coherent sheaf over $\mathcal{X}$ flat over S. If $R^i q_*(\mathcal{F}) = (0)$ for $i \geq i_0$ then $H^i(\mathcal{X}_s, \mathcal{F}_s) = (0)$ for any $s \in S$ and $i \geq i_0$.*

Corollary B.10. [*The SeeSaw Principle*].
Let $\mathcal{X} \to X$ be a good moduli space over an integral scheme S and $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Suppose that $q : \mathcal{X} \to S$ is flat and that $X \to S$ is proper with integral geometric fibers. Then the locus

$$S_1 = \{s \in S | \mathcal{L}_s \cong \mathcal{O}_{\mathscr{X}_s}\}$$

is closed in S. Moreover if we call $q_1 : \mathcal{X} \times_S S_1 \to S_1$ the restriction of q on this locus, then $q_{1*}\mathcal{L}$ is a line bundle on S and the natural morphism $q_1^* q_{1*}\mathcal{L} \cong \mathcal{L}$ is an isomorphism.

Proof. A line bundle $\mathcal{M}$ on a stack $\mathcal{X}$ with a proper integral good moduli space X is trival if and only if $h^0(\mathcal{M}) > 0$ and $h^0(\mathcal{M}^{-1}) > 0$. The necessity is obvious. Conversely suppose that these conditions hold. Then we have two non-zero homomorphisms $s : \mathcal{O}_{\mathcal{X}} \to \mathcal{M}$, $t : \mathcal{O}_{\mathcal{X}} \to \mathcal{M}^{-1}$. If we dualize the second one and compose with the first one, we have a non-zero morphism $h : \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{X}}$. Now X is an integral proper scheme then $H^0(X, \mathcal{O}_X) = k$ so $H^0(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) = k$. Hence h is an isomorphism. This implies that also s and t are isomorphisms. As a consequence, we have

$$S_1 = \{s \in S | h^0(\mathcal{X}_s, \mathcal{L}_s) > 0,\, h^0(\mathcal{X}_s, \mathcal{L}_s^{-1}) > 0\}.$$

In particular, S_1 is closed by upper semicontinuity. Up to restriction we can assume $S = S_1$, so the function $s \mapsto h^0(\mathcal{X}_s, \mathcal{L}_s) = 1$ is constant. By Corollary B.7, $q_*\mathcal{L}$ is a line bundle on S and the natural map $q_*\mathcal{L} \otimes_{\mathcal{O}_S} k(s) \to H^0(\mathcal{X}_s, \mathcal{L}_s)$ is an isomorphism. Consider the natural map $\pi : q^* q_* \mathcal{L} \to \mathcal{L}$. Its restriction on any fiber $\mathcal{X}_s$

$$\mathcal{O}_{\mathcal{X}_s} \otimes H^0(\mathcal{X}_s, \mathcal{L}_s) \to \mathcal{L}_s$$

is an isomorphism. In particular π is an isomorphism for any geometric point $x \in \mathcal{X}$. Since it is a map between line bundles, by Nakayama lemma, it is an isomorphism. $\square$

References

[AC87] Enrico Arbarello and Maurizio Cornalba. The Picard groups of the moduli spaces of curves. *Topology*, 26(2):153–171, 1987.

[ACG11] Enrico Arbarello, Maurizio Cornalba, and Pillip A. Griffiths. *Geometry of algebraic curves. Volume II*, volume 268 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011. With a contribution by Joseph Daniel Harris.

[ACV03] Dan Abramovich, Alessio Corti, and Angelo Vistoli. Twisted bundles and admissible covers. *Comm. Algebra*, 31(8):3547–3618, 2003. Special issue in honor of Steven L. Kleiman.

[Alp10] Jarod Alper. On the local quotient structure of Artin stacks. *J. Pure Appl. Algebra*, 214(9):1576–1591, 2010.

[Alp13] Jarod Alper. Good moduli spaces for Artin stacks. *Ann. Inst. Fourier (Grenoble)*, 63(6):2349–2402, 2013.

[BFMV14] Gilberto Bini, Fabio Felici, Margarida Melo, and Filippo Viviani. *Geometric invariant theory for polarized curves*, volume 2122 of *Lecture Notes in Mathematics*. Springer, Cham, 2014.

[BH12] Indranil Biswas and Norbert Hoffmann. Poincaré families of G-bundles on a curve. *Math. Ann.*, 352(1):133–154, 2012.

[Bro] Syilvain Brochard. *Champs algébriques et foncteur de Picard. PhD thesis*. Available at http://www.math.univ-montp2.fr/ brochard/.

[Bro12] Sylvain Brochard. Finiteness theorems for the Picard objects of an algebraic stack. *Adv. Math.*, 229(3):1555–1585, 2012.

[Cap94] Lucia Caporaso. A compactification of the universal Picard variety over the moduli space of stable curves. *J. Amer. Math. Soc.*, 7(3):589–660, 1994.

[CMKV15] Sebastian Casalaina-Martin, Jesse Leo Kass, and Filippo Viviani. The local structure of compactified Jacobians. *Proc. Lond. Math. Soc. (3)*, 110(2):510–542, 2015.

[Cor07] Maurizio Cornalba. The Picard group of the moduli stack of stable hyperelliptic curves. *Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl.*, 18(1):109–115, 2007.

[DM69] P. Deligne and D. Mumford. The irreducibility of the space of curves of given genus. *Inst. Hautes Études Sci. Publ. Math.*, (36):75–109, 1969.

[Edi13] Dan Edidin. Equivariant geometry and the cohomology of the moduli space of curves. In *Handbook of moduli. Vol. I*, volume 24 of *Adv. Lect. Math. (ALM)*, pages 259–292. Int. Press, Somerville, MA, 2013.

[EG98] Dan Edidin and William Graham. Equivariant intersection theory. *Invent. Math.*, 131(3):595–634, 1998.

[Fal96] Gerd Faltings. Moduli-stacks for bundles on semistable curves. *Math. Ann.*, 304(3):489–515, 1996.

[FGI$^+$05] Barbara Fantechi, Lothar Göttsche, Luc Illusie, Steven L. Kleiman, Nitin Nitsure, and Angelo Vistoli. *Fundamental algebraic geometry*, volume 123 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2005. Grothendieck's FGA explained.

[Gie82] D. Gieseker. *Lectures on moduli of curves*, volume 69 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Published for the Tata Institute of Fundamental Research, Bombay; Springer-Verlag, Berlin-New York, 1982.

[Gie84] D. Gieseker. A degeneration of the moduli space of stable bundles. *J. Differential Geom.*, 19(1):173–206, 1984.

[Gir71] Jean Giraud. *Cohomologie non abélienne*. Springer-Verlag, Berlin-New York, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

[Gri] Matt Grimes. *Universal moduli spaces of vector bundles and the log-minimal model program on the moduli of curves*. Available at http://arxiv.org/abs/1409.5734.

[GV08] Sergey Gorchinskiy and Filippo Viviani. Picard group of moduli of hyperelliptic curves. *Math. Z.*, 258(2):319–331, 2008.

[Har83] John Harer. The second homology group of the mapping class group of an orientable surface. *Invent. Math.*, 72(2):221–239, 1983.

[HL10] Daniel Huybrechts and Manfred Lehn. *The geometry of moduli spaces of sheaves*. Cambridge Mathematical Library. Cambridge University Press, Cambridge, second edition, 2010.

[HM98] Joe Harris and Ian Morrison. *Moduli of curves*, volume 187 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1998.

[Hof07] Norbert Hoffmann. Rationality and Poincaré families for vector bundles with extra structure on a curve. *Int. Math. Res. Not. IMRN*, (3):Art. ID rnm010, 30, 2007.

[Hof10] Norbert Hoffmann. Moduli stacks of vector bundles on curves and the King-Schofield rationality proof. In *Cohomological and geometric approaches to rationality problems*, volume 282 of *Progr. Math.*, pages 133–148. Birkhäuser Boston, Inc., Boston, MA, 2010.

[Hof12] Norbert Hoffmann. The Picard group of a coarse moduli space of vector bundles in positive characteristic. *Cent. Eur. J. Math.*, 10(4):1306–1313, 2012.

[Kau05] Ivan Kausz. A Gieseker type degeneration of moduli stacks of vector bundles on curves. *Trans. Amer. Math. Soc.*, 357(12):4897–4955 (electronic), 2005.

[Kou91] Alexis Kouvidakis. The Picard group of the universal Picard varieties over the moduli space of curves. *J. Differential Geom.*, 34(3):839–850, 1991.

[Kou93] Alexis Kouvidakis. On the moduli space of vector bundles on the fibers of the universal curve. *J. Differential Geom.*, 37(3):505–522, 1993.

[LMB00] Gérard Laumon and Laurent Moret-Bailly. *Champs algébriques*, volume 39 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2000.

[Mel09] Margarida Melo. Compactified picard stacks over $\overline{\Updownarrow}_g$. *Math. Z.*, 263(4):939–957, 2009.

[MFK94] D. Mumford, J. Fogarty, and F. Kirwan. *Geometric invariant theory*, volume 34 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (2) [Results in Mathematics and Related Areas (2)]*. Springer-Verlag, Berlin, third edition, 1994.

[MR85] N. Mestrano and S. Ramanan. Poincaré bundles for families of curves. *J. Reine Angew. Math.*, 362:169–178, 1985.

[Mum65] David Mumford. Picard groups of moduli problems. In *Arithmetical Algebraic Geometry (Proc. Conf. Purdue Univ., 1963)*, pages 33–81. Harper & Row, New York, 1965.

[Mum70] David Mumford. *Abelian varieties*. Tata Institute of Fundamental Research Studies in Mathematics, No. 5. Published for the Tata Institute of Fundamental Research, Bombay; Oxford University Press, London, 1970.

[Mum83] David Mumford. Towards an enumerative geometry of the moduli space of curves. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 271–328. Birkhäuser Boston, Boston, MA, 1983.

[MV14] Margarida Melo and Filippo Viviani. The Picard group of the compactified universal Jacobian. *Doc. Math.*, 19:457–507, 2014.

[NS99] D. S. Nagaraj and C. S. Seshadri. Degenerations of the moduli spaces of vector bundles on curves. II. Generalized Gieseker moduli spaces. *Proc. Indian Acad. Sci. Math. Sci.*, 109(2):165–201, 1999.

[Pan96] Rahul Pandharipande. A compactification over $\overline{M}_g$ of the universal moduli space of slope-semistable vector bundles. *J. Amer. Math. Soc.*, 9(2):425–471, 1996.

[Sch04] Alexander Schmitt. The Hilbert compactification of the universal moduli space of semistable vector bundles over smooth curves. *J. Differential Geom.*, 66(2):169–209, 2004.

[Ser06] Edoardo Sernesi. *Deformations of algebraic schemes*, volume 334 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

[Ses82] C. S. Seshadri. *Fibrés vectoriels sur les courbes algébriques*, volume 96 of *Astérisque*. Société Mathématique de France, Paris, 1982. Notes written by J.-M. Drezet from a course at the École Normale Supérieure, June 1980.

[TiB91] Montserrat Teixidor i Bigas. Brill-Noether theory for stable vector bundles. *Duke Math. J.*, 62(2):385–400, 1991.

[TiB95] Montserrat Teixidor i Bigas. Moduli spaces of vector bundles on reducible curves. *Amer. J. Math.*, 117(1):125–139, 1995.

[TiB98] Montserrat Teixidor i Bigas. Compactifications of moduli spaces of (semi)stable bundles on singular curves: two points of view. *Collect. Math.*, 49(2-3):527–548, 1998. Dedicated to the memory of Fernando Serrano.

[Wan] Jonathan Wang. The moduli stack of $\mathfrak{g}$-bundles. Available at http://arxiv.org/abs/1104.4828.